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An algebraic approach to configuration spaces.

M n-manifold.

$\mathrm{Conf}_k(M)$ = k-tuples of distinct points in M.

$B_k(M) = \mathrm{Conf}_k(M)_{\Sigma_k}$.

Ex. $B_2(L_{7,1}) \simeq B_2(L_{7,2})$

Longoni–Salvatore.

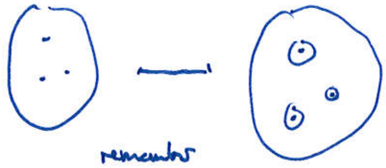
Some lens spaces. $L_{7,1}$ and $L_{7,2}$ are homotopy $\simeq$ but not homeomorphic.

These are complicated, but less so over $\mathbb{Q}$.

Segal, McDuff, Cohen, Bödigheimer, ...

Goals. Unify. Extend.

Key observations. * $\mathrm{Conf}_k(\mathbb{R}^n) \xleftarrow{\sim} \mathrm{Emb}^{fr}\left(\coprod_k \mathbb{R}^n, \mathbb{R}^n\right) =: \mathbb{E}_n(k)$



remember centers.

$\{E_n(k)\}$ is an $\mathbb{E}_n$-operad and $B(\mathbb{R}^n) = \coprod B_k(\mathbb{R}^n)$ is an algebra over $\mathbb{E}_n$.

* Conf_k is "multi-local": $\{\mathrm{Conf}_k(U) \mid U \subseteq M_{\text{open}},\ U \cong \coprod_k \mathbb{R}^n\}$ is a basis for topology of $\mathrm{Conf}_k(M)$.

Globalization (factorization homology).

A an $\mathbb{E}_n$-algebra in $\mathrm{Ch}_{\mathbb{Q}}$.

$$\coprod_k \mathbb{R}^n \xrightarrow[\text{framed}]{} \coprod_l \mathbb{R}^n \rightsquigarrow A^{\otimes k} \to A^{\otimes l}$$

$$(\mathrm{Disk}_n^{fr}, \amalg) \xrightarrow{A} (\mathrm{Ch}_{\mathbb{Q}}, \otimes)$$

Ex. $B_k(\mathbb{R}^n \amalg \mathbb{R}^n) \cong \coprod_{i+j=k} B_i(\mathbb{R}^n) \times B_j(\mathbb{R}^n)$

$$C_*(B(\mathbb{R}^n \amalg \mathbb{R}^n)) \simeq C_*(B(\mathbb{R}^n))^{\otimes 2}$$

$$\begin{tikzcd}
(\mathrm{Disc}_n^{fr}, \amalg) \arrow[r, "A"', "\text{symmetric monoidal}"] \arrow[d, hook] & (\mathrm{Ch}_{\mathbb{Q}}, \otimes) \\
(\mathrm{Mfld}_n^{fr}, \amalg) \arrow[ur, dashed, "\text{left Kan}", "\int_- A"']
\end{tikzcd}$$

Thm (Francis). $\int_- A$ is uniquely specified by

- $\int_- A|_{\mathrm{Disc}_n^{fr}} \simeq A$ as $\mathbb{E}_n$-algebra.
- If $M \cong M_1 \amalg_{M_0 \times \mathbb{R}} M_2 \Rightarrow \int_M A \simeq \int_{M_1} A \otimes^{\mathbb{L}}_{\int_{M_0 \times \mathbb{R}} A} \int_{M_2} A$ "excision."

Can also do this for disc-n algebras.

Since $\mathrm{Conf}_2(\mathbb{R}^n) \simeq S^{n-1}$, an $\mathbb{E}_n$-algebra has two binary ops, corresponding to the 2 cells of S^{n-1}.

$m_0 : A^{\otimes 2} \to A$ (deg 0)

$m_{n-1} : A^{\otimes 2} \to A[1-n]$ shifted Lie.

Thm (Cohen). $H_*(\mathbb{E}_n) \simeq \mathrm{Pois}_n$.

Thm (Fresse). There is an essentially unique map of operads

$$\Lambda^{1-n} L_\infty \to \mathbb{E}_n .$$

$$\begin{tikzcd} \mathrm{Alg}_{\mathrm{Lie}} \arrow[rr, bend right, "U_n"'] & \mathrm{Alg}_{\mathrm{Lie}_{n}} \arrow[l, "\text{shift}"'] & \mathrm{Alg}_{\mathbb{E}_n} \arrow[l] \end{tikzcd}$$

$\mathbb{E}_n$ enveloping algebra.

Hence, $C_*(B(-)) \simeq U_n(\mathrm{Lie}(\mathbb{Q}[n-1]))$ ★
since $B(-)$ is the free $\mathbb{E}_n$-algebra on a pt.

Apply $\int_M$ to ★:

LHS is $C_*(B(M))$ (Lurie's van Kampen theorem),

RHS $\simeq C^{\mathrm{Lie}}_\bullet(\Omega_c(M, \mathrm{Lie}(\mathbb{Q}[n-1])))$.

compact de Rham.

Recall: $C^{\mathrm{Lie}}(\mathfrak{g}) \simeq \mathbb{Q} \otimes^L_{U\mathfrak{g}} \mathbb{Q}$

$\simeq \mathrm{Sym}(\mathfrak{g}[1])$

Chevalley-Eilenberg.

Turns out to be formal!

Only for M framed. Worse, the natural n-disc algebra refinements disagree.

Splitting points.

$$\mathrm{Conf}_k \longrightarrow \mathrm{Conf}_i \times \mathrm{Conf}_j$$

Apply C_*

$$\bigoplus_k \mathrm{Conf}_k \longrightarrow \bigoplus_k \bigoplus_{i+j=k} \mathrm{Conf}(\mathrm{Conf}_i \times \mathrm{Conf}_j) \quad \underset{\text{symmetrize}}{\rightsquigarrow} \quad C_*(B) \to C_*(B)^{\otimes 2}$$

$$C_*(\mathrm{Conf}_\bullet) \xrightarrow[\text{coalgebra}]{} C_*(\mathrm{Conf})^{\otimes 2}$$

So, $C_*(B)$ is an n-disk algebra in cocommutative coalgebras.

This is the definition of an n-Hopf algebra.

Analog of Milnor-Moore. Thm (in progress). If A is an n-Hopf algebra, then $\mathrm{Prim}(A)$ is an n-disk algebra in Lie algebras, and

$$C^{\mathrm{Lie}}(\mathrm{Prim}(A)) \xrightarrow{\sim} A$$

as n-disk algebras.

proof outline. (1) PBW.

$\mathbb{E}_1 \to \mathrm{Disk}_n \Rightarrow A$ is a monoid
$\to$ primitive filtration splits.

Barnes–McCarthy Goodwillie calculus.

(2) $(C^{\mathrm{Lie}}, \mathrm{Prim})$ adjunction is well-behaved on cofree coalgebras.

Thm (K.) $B^n \mathrm{Prim}(C_*(B)) \simeq \mathrm{Lie}(\mathbb{Q}^{\det}[n-1])$.

Thm (K). There is an iso. of bigraded coalgebras

$$H_*(B(M)) \cong H_*^{Lie}\left(H_c^{-*}\left(M; Lie\left(\mathbb{Q}^w[n-1]\right)\right)\right)$$

coming from qis.

orientation sheaf

Formality!

Follows from simplicity of $Lie(\mathbb{Q}^w[n-1])$.

Cor. $H_*(B_k(M))$ depends only on

* $H_*(M)$ if n is odd
* $H_c^{-*}(M, \mathbb{Q}^w)^{\otimes 2} \longrightarrow H_c^{-*}(M, \mathbb{Q})$

  n even.

Generalizes various results ... Félix-Thomas.

Thm (K). Let M be connected, ns1.

$$1_M \cap -: H_*(B_{k+1}(M)) \longrightarrow H_*(B_k(M))$$

is an iso. for

- $\deg < k$ if M is an orientable surface,
- $\le k$ otherwise.

Union of theorems of Church, Randal-Williams.