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A theory of elementary higher toposes.
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1) Grothendieck topos.

2) Elementary topos.

3) Higher topos.

4) Complete Segal objects.

5) Elementary higher topos.

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1) Grothendieck topos.

Need more points in alg. geometry.

Def. A <u>Grothendieck topos</u> is a left exact localization of presheaves of sets on a small cat. $\mathcal{C}$.

$$\mathrm{Fun}(\mathcal{C}^{op}, Sets) \underset{a}{\overset{\text{fully faithful}}{\rightleftarrows}} \mathcal{G} \quad \text{topos}$$

left exact, preserves finite limits.

a is sheafification.

Ex. X a topological space.

$Open_X$ category of open sets and inclusions.

$$\mathrm{Fun}(Open_X^{op}, Sets) \longleftarrow \mathcal{G}.$$

Say $\mathcal{F} \in \mathcal{G}$ if and only if for every open cover, $U = \bigcup_{i \in I} U_i$,

$$\mathcal{F}(U) \to \prod_{i \in I} \mathcal{F}(U_i) \rightrightarrows \prod_{(i,j) \in I^2} \mathcal{F}(U_i \cap U_j)$$

is an equalizer.

It turns out that $\mathcal{G}$ is a left exact loc, so a topos.

2) Elementary topos.

Lawvere.

Tierney.

<u>Def</u>. An <u>elementary topos</u> is a loc. cartesian closed category with a subobject classifier.

<u>Def</u>. A category $\mathcal{C}$ is <u>cartesian</u> <u>closed</u> if it has finite products (and in particular a final object) and if for all $c \in \mathcal{C}$ if there is a right adjoint

$$\mathcal{C} \underset{(-)\times c}{\overset{(-)^c}{\rightleftarrows}} \mathcal{C}.$$

<u>Def</u>. A category $\mathcal{C}$ is <u>locally</u> <u>cartesian</u> <u>closed</u> if for all $c \in \mathcal{C}$ the category $\mathcal{C}_{/c}$ is cartesian closed.

<u>Def</u>. If $\mathcal{C}$ has finite limits, then

$$\mathrm{Sub}(-) : \mathcal{C}^{op} \to \mathrm{Set},$$

$$\mathrm{Sub}(c) = \{ d \to c \text{ a categorical mono} \} / \text{iso}.$$

<u>Def</u>. A subobject classifier Ω is an object that represents $\mathrm{Sub}(-)$. So, we have an iso $\mathrm{Sub}(-) \cong \mathrm{Hom}_{\mathcal{C}}(-, \Omega)$.

<u>Ex</u>. In Sets, we can take $\Omega = \{0,1\}$. Take the characteristic function of a subset.

<u>Ex</u>. Every Grothendieck topos is an elementary topos. A presentable elementary topos is a Grothendieck topos.

3) Higher topos theory.

Lurie.

Def. A higher topos $\mathcal{G}$ is an accessible left exact localization of an $(\infty,1)$-cat of presheaves of spaces on a small $(\infty,1)$-cat $\mathcal{C}$.

$$\begin{tikzcd} \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Spaces}) \arrow[r, shift right, "a"'] & \mathcal{G} \arrow[l, shift right, "\text{f.f.}"'] \end{tikzcd}$$

↑ preserves finite limits.

Ex. Spaces. Whatever an elementary topos is, $\mathcal{S}$, the cat. of spaces should be an ex. It is loc. cart. closed and $\Omega = \{0,1\}$ is a subobject classifier.

Spaces = small.

SPACES = big.

$\mathcal{C} \underset{\text{core}}{\rightsquigarrow}$ maximal subgroupoid $\in$ SPACES

$(\mathrm{Spaces})^{\text{core}} \in \mathrm{SPACES}$.

$$\begin{tikzcd} \mathrm{Pb} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & E \arrow[d] \\ \mathcal{C} \arrow[r] & \mathrm{Spaces}^{\text{core}} \end{tikzcd}$$

← fiber over $X \in$ Spaces is X.

Intuitively, SPACES has an object classifier.

~~[illegible]~~

Rem. From now on, higher cats $\mathcal{C}$ always have finite limits.

Def. There is a functor

$$\begin{array}{rcl} \mathcal{C}_{/-} : \mathcal{C}^{op} & \longrightarrow & Cat_\infty \\ c & \longmapsto & \mathcal{C}_{/c} \\ (c \to d) & \longmapsto & \mathcal{C}_{/d} \longrightarrow \mathcal{C}_{/c} \end{array}$$

pullback along $c \to d$.

Def. U is an object classifier if there is an embedding $y_U \hookrightarrow (\mathcal{C}_{/-})^{core}$.

$$\begin{tikzcd} (Spans)^{core} \\ (Spans^{[1]})^{core} \arrow[u, shift left=2, "t"'] \arrow[u, shift right=2, "s"] \\ (Spans^{[2]})^{core} \arrow[u, shift left=3] \arrow[u] \arrow[u, shift right=3] \\ \vdots \end{tikzcd}$$

A simplicial SPACE.

4) Complete Segal spaces.

There are models of higher cats called complete Segal spaces.

Def. A <u>complete Segal space</u> is a simplicial space $\Delta^{op} \to S$ satisfying some conditions.

These conditions only require finite limits.
So, we can define <u>complete Segal objects in $\mathcal{C}$</u> (with finite limits).

Internal higher cat object.

Get $\mathcal{X}_\bullet : \Delta^{op} \to \mathcal{C}$.

~~[illegible]~~ Get $\mathcal{C}^{op} \longrightarrow \mathcal{C}at_\infty$

$c \xmapsto[\mathcal{Y}_{\mathcal{X}_\bullet}]{} map(c, \mathcal{X}_\bullet)$, a complete Segal space.

<u>Thm</u> (Rasekh). <u>Representable</u> <u>Cartesian</u> <u>Fibrations</u> let us do this rigorously, get $\mathcal{Y}_{\mathcal{X}}$.

Def. A presheaf $F : \mathcal{C}^{op} \longrightarrow \mathcal{C}at_\Delta$ is representable if it is equivalent to $\mathcal{Y}_{\mathcal{X}_\bullet}$ for some complete Segal object $\mathcal{X}_\bullet$ in $\mathcal{C}$.

5) Elementary higher toposes.

Def. An elementary higher topos is an $(\infty,1)$-cat $\mathcal{E}$ s.t.

1) $\mathcal{E}$ has finite limits & colimits,
2) $\mathcal{E}$ has a subobject classifier,
3) for every map $f \in \mathcal{E}$, $f: d \to c$, there exists $\mathcal{U}_\bullet \in CSO(\mathcal{E})$ s.t. $\mathcal{Y}_{\mathcal{U}_\bullet} \hookrightarrow \mathcal{E}_{/c}$ f is in the image of embedding.

Why do we care?

1) HTT still not well-understood.

2) Classify localizations.

3) Understand different notions of $\mathcal{S}$.

4) Homotopy type theory.