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Norms in motivic homotopy theory.

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joint w/ Tom Bachmann.

<u>Motivation</u>. (1) Norms in <u>equivariant</u> <u>homotopy</u> <u>theory</u> from Hill-Hopkins-Ravenel. $H \le G$. $N_H^G : Sp^H \to Sp^G$. Not left or right adjoints. So, not a Quillen functor. Notion of a G-$\mathbb{E}_\infty$-ring spectrum. $\mathbb{E}_\infty$ in Sp^G with norms.

n-excisive when $n = [H:G]$.

<u>Ex</u>. K-theory. G action on X, G finite. $H \le G$.

$$N_H^G : K_0^H(X) \to K_0^G(X).$$

Get them on K-theory spectra, $\mathbb{E}_\infty$ for the multiplicative structure.

$V \to X$ a G-vector bundle

$\mapsto \bigotimes_{G/V} V$

Norms on Sp^G are a categorification of this example.

(2) Norms in <u>algebraic</u> <u>geometry</u>. $p : Y \to X$ finite étale. Fulton-Macpherson.

$$N_p : CH^*(Y) \to CH^*(X) \qquad (X/k \text{ sm. quasi-projective}).$$

If $p = \nabla : X \amalg X \to X$,

then N_∇ is the usual multiplication.

Joukhovitski. $N_p : K_0(Y) \to K_0(X)$. Something like $\otimes$-integration over fibers.

Comes from an excisive norm functor $Perf_Y \to Perf_X$.

These are all <u>Tambara</u> <u>functors</u>.

Goals: (a) extend these norms to higher Chow groups/higher K-theory,
(b) get norm functors in motivic homotopy theory.
(c) compare with equivariant picture for étale fundamental groups.

<u>Warmup</u>. Norms in ordinary cohomology.

$p : Y \longrightarrow X$ finite cov. map of top. spaces of degree d.

$$N_p : H^n(Y) \longrightarrow H^{nd}(X).$$

If $Y = \underbrace{X \amalg \cdots \amalg X}_{d}$ and p is the fold map, N_p is d-fold cup product.

Idea: work locally. Choose a covering $U \to X$ that splits p.

$$U \times_X Y \simeq U^{\amalg d}.$$

$$\begin{array}{ccc} C^\cdot(X) \simeq \mathrm{Tot}\big(& C^\cdot(U) \rightrightarrows & C^\cdot(U \times_X U) \substack{\to\\\to\\\to} \cdots \big) \\ & \uparrow \cup & \uparrow \cup \\ C^\cdot(Y) \simeq \mathrm{Tot}\big(& C^\cdot(U)^{d} \rightrightarrows & C^\cdot(U \times_X U)^{d} \substack{\to\\\to\\\to} \cdots \big) \end{array}$$

But, $\cup$ is not linear, so this is nonsense.

Also, cannot use the underlying space, since that kills all the cohomology.

Replace $C^\cdot(U)^d$ with $C^\cdot(U)^{\otimes d}$

$$\mathrm{Map}(U, H\mathbb{Z}^{\wedge d})$$

Limit $C^\cdot(Y)$ with $\Gamma_X\big(\bigwedge_{Y/X} H\mathbb{Z}_Y\big)$.

$$\begin{array}{c} \Gamma_X\big(\bigwedge_{Y/X} H\mathbb{Z}_Y\big) \\ | \\ \Gamma_X(H\mathbb{Z}_X) \end{array}$$

<u>Recipe for $\bigwedge_{Y/X}$</u>.
Norm functor in parametrized homotopy theory.

$$\bigwedge_{Y/X}(U) = \mathrm{Tot}\Big(C^\cdot(U)^{\otimes d} \rightrightarrows C^\cdot(U \times_X U)^{\otimes d} \substack{\to\\\to\\\to} \Big)$$

$$H^n(Y) = [S^n_Y, H\mathbb{Z}_Y] \xrightarrow{\bigwedge_{Y/X}} \Big[\bigwedge_{Y/X} S^n_Y, \bigwedge_{Y/X} H\mathbb{Z}_Y\Big] \longrightarrow \Big[\bigwedge_{Y/X} S^n_Y, H\mathbb{Z}_X\Big]$$

Thom $\simeq H^{nd}(X)$.

Symmetric monoidal structures.

$(\mathcal{C}, \otimes, \mathbb{1})$ $\quad \otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$. Some axioms.

Unbiased version. $p : I \to J$ of finite sets.

$$p_\otimes : \mathcal{C}^I \to \mathcal{C}^J.$$

$I \mapsto \mathcal{C}^I$ contravariant

$$\begin{tikzcd} \mathrm{Fin}^{op} \arrow[r] \arrow[d, hook] & \mathrm{Cat} \\ \mathrm{Span}(\mathrm{Fin}) \arrow[ur, dashed] & \end{tikzcd}$$

This extension is $\simeq$ to giving a sym. mon. structure on $\mathcal{C}$.

$$I \xleftarrow{f} J \xrightarrow{p} K$$

Usual pullbacks.

Fin and Fin^{op} inside.

Also works for Cat_∞.

We can easily define $\mathbb{E}_\infty$-algebras in this setting.

$I \in \mathrm{Fin}$, $\quad A_I \in \mathcal{C}^I$.

$f^* A_I \xrightarrow{\sim} A_J \qquad$ Determined by $A_{\{*\}}$.

$p_\otimes A_J \to A_K \qquad$ Multiplication.

Section of the cocartesian fibration.

Equivariant homotopy theory.

$$\begin{tikzcd}
\mathrm{FinGpd}^{op} \arrow[r] \arrow[d, hook] & \mathrm{Cat}_\infty \\
\mathrm{Span}_{fold}(\mathrm{FinGpd}) \arrow[ur, dashed] \arrow[d, hook] & \\
\mathrm{Span}_{fin.cov.}(\mathrm{FinGpd}) \arrow[uur, dashed] &
\end{tikzcd}$$

$BG \longmapsto Sp^G$

These are symmetric monoidal.

This is the symmetric monoidal enrichment $\mathrm{FinGpd}^{op} \longrightarrow \mathrm{SymMon}_\infty$.

$$\begin{tikzcd}
 & J \arrow[dl] \arrow[dr, "\mathrm{Fold}"] & \\
I & & K
\end{tikzcd}$$

Norms.

$N_H^G := p_\otimes$

$BH \xrightarrow{p} BG.$

Motivic version.

$X \longmapsto SH(X) \simeq \mathrm{Pre}_*(Sm/k)_{\mathbb{A}^1,\mathrm{Nis}}[(\mathbb{P}^1,\infty)^{-1}]$

$\otimes$-localization.

$$\begin{tikzcd}
\mathrm{Sch}^{op} \arrow[r] \arrow[d, hook] & \mathrm{Cat}_\infty \\
\mathrm{Span}_{fold}(\mathrm{Sch}) \arrow[ur, dashed] \arrow[d, hook] & \\
\mathrm{Span}_{fin.\acute{e}t}(\mathrm{Sch}) \arrow[uur, dashed] &
\end{tikzcd}$$

Presheaf of sym monoidal ∞-cats

Norms.

$p: Y \longrightarrow X$ fin. ét

$p_\otimes : SH(Y) \longrightarrow SH(X).$

Variants.

(a) $\mathrm{DM}(X)$.

Categorification of Chow groups.

$$CH^n(X) \cong \pi_0 \operatorname{Map}_{\mathrm{DM}(X)}(\mathbb{1}, \mathbb{1}(n)[2n]).$$

(b) $\mathrm{NcMot}(X)$. À la Robalo.

$$K_i(X) \cong \pi_i \operatorname{Map}(\mathbb{1}, \mathbb{1}).$$

Get normed $\mathbb{E}_\infty$-algebras.

Thm. $H\mathbb{Z}$, KGL are normed motivic spectra.
So is MGL.

Related to Witt groups.

Non-ex. $\mathbb{S}[\eta^{-1}]$ Hopf-inverted motivic spectra.
This is $\mathbb{E}_\infty$ but not normed. Same with $\mathbb{S}[p^{-1}]$.

Rem. Get normed cyclotomic $\mathbb{E}_\infty$-rings for the étale fundamental groups.