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Norms in motivic homotopy theory.

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joint w/ Tom Bachmann.

Motivation.

(1) Norms in equivariant homotopy theory from

Hill-Hopkins-Ravenel.  $H \leq G$ .  $N_H^G: Sp^H \rightarrow Sp^G$ .

Not left or right adjoint. So, not a Quillen functor.

Notion of a  $G$ -Eilenberg-MacLane spectrum.  $E_n$  in  $Sp^G$  with norms.

$n$ -excision when  $n = [H:G]$ .

Ex. K-theory.  $G$  action on  $X$ ,  $G$  finite.  $H \leq G$ .

$$N_H^G: K_0^H(X) \rightarrow K_0^G(X).$$

Get them on K-theory spans,  $E_n$  for the multiplication structure.

$V \rightarrow X$  a  $G$ -vector bundle

$$\rightarrow \begin{matrix} \square & \otimes & V \\ G/V & & \end{matrix}$$

Norms on  $Sp^G$  as a categorification of this example.

(2) Norms in algebraic geometry.  $p: Y \rightarrow X$  finite étale.

Fulton-Macpherson.

$$N_p: CH^*(Y) \rightarrow CH^*(X) \quad (X/\mathbb{k} \text{ sm. quasi-projective}).$$

If  $p = \nabla: X \amalg X \rightarrow X$ ,

then  $N_p$  is the usual multiplication.

Jonkhartskii.  $N_p: K_0(Y) \rightarrow K_0(X)$ . Something like  $\otimes$ -integration over fibers.

These are all Tambara functors.

Comes from an excision norm functor  $Perf_Y \rightarrow Perf_X$ .

- Goals: (a) extend these norms to higher Chow groups/higher K-theory,  
 (b) get norm functors in motivic homotopy theory.  
 (c) compare with equivariant points for étale fundamental groups.

Warmup. Norms in ordinary cohomology.

$p: Y \longrightarrow X$  finite cov. map  
 of top. spaces of degree  $d$ .

$$N_p: H^n(Y) \longrightarrow H^{nd}(X).$$

If  $Y = \underbrace{X \amalg \dots \amalg X}_d$  and  $p$  is the fold map,  $N_p$  is  $d$ -fold cup product.

Idea: work locally. Choose a covering  $U \rightarrow X$  that splits  $p$ .

$$U \times_X Y \cong U^{\amalg d}.$$

$$\begin{array}{c} C^*(X) \cong \text{Tot} (C^*(U) \rightrightarrows C^*(U \times_X U) \rightrightarrows \dots) \\ \uparrow \quad \quad \quad \uparrow \\ C^*(Y) \cong \text{Tot} (C^*(U)^{\amalg d} \rightrightarrows C^*(U \times_X U)^{\amalg d} \rightrightarrows \dots) \end{array}$$

But,  $\cup$  is not linear, so this is nonsense.

Replace  $C^*(U)^{\amalg d}$  with  $C^*(U)^{\odot d}$

Also, cannot use the underlying space, since that kills all the cohomology.

$$\Gamma_{op}(U, H\mathbb{Z}^{nd})$$

imitate  $C^*(Y)$  with  $\Gamma_X(\bigwedge_{Y/X} H\mathbb{Z}_Y)$ .

$$\Gamma_X(H\mathbb{Z}_X)$$

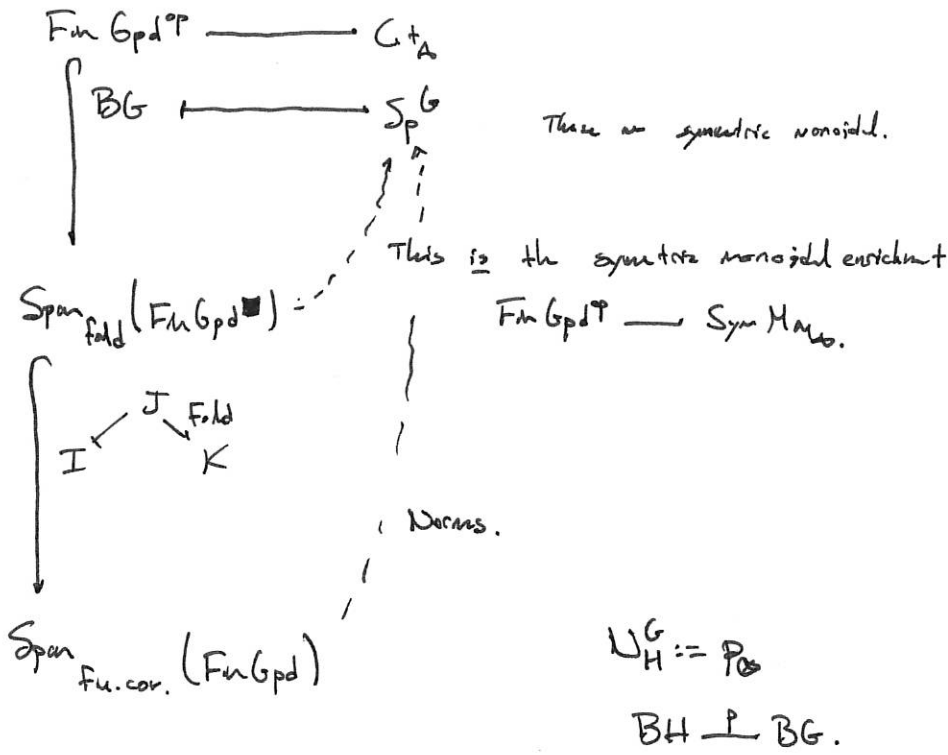
Recipe for  $\Lambda_{Y/X}$ .  
 Norm functor in parametrized homotopy theory.  
 $\Lambda_{Y/X}(U) = \text{Tot} (C^*(U)^{\odot d} \rightrightarrows C^*(U \times_X U)^{\odot d} \rightrightarrows \dots)$

$$H^n(Y) = [\Sigma^n_Y, H\mathbb{Z}_Y] \xrightarrow{\Lambda_{Y/X}} [\bigwedge_{Y/X} S^{-n}_Y, \bigwedge_{Y/X} H\mathbb{Z}_Y] \longrightarrow [\bigwedge_{Y/X} S^{-n}_X, H\mathbb{Z}_X]$$

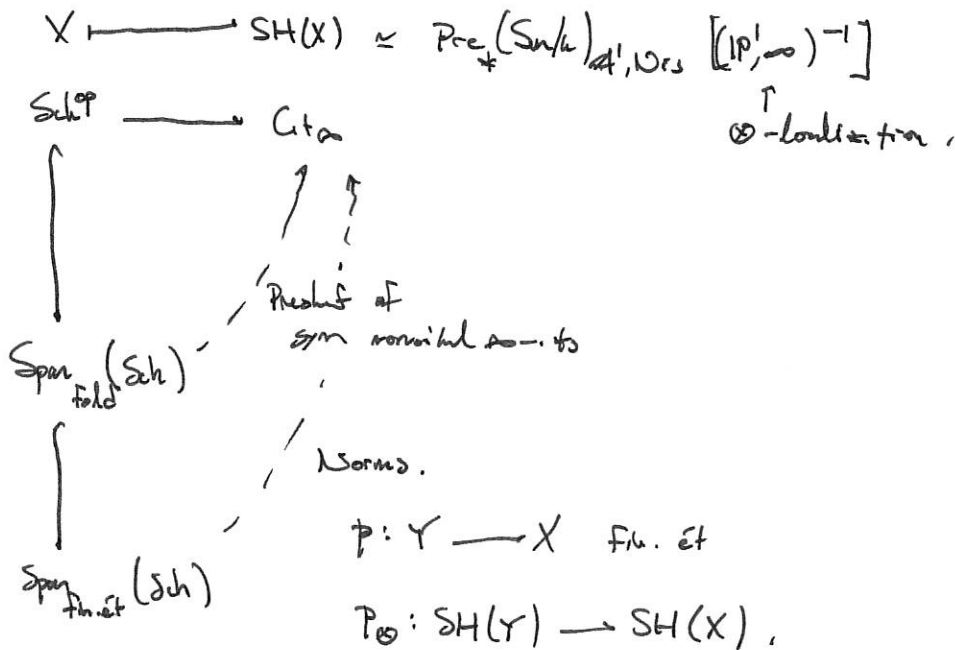
Then  $\cong H^{nd}(X)$ .



Equivalent homotopy theory.



Motivic version.



Variants.

(a)  $DM(X)$ .

Categorification of Chow groups.

$$CH^1(X) \cong \pi_0 \Gamma_{\mathbb{P}_{DM(X)}}(\mathbb{1}, \mathbb{1}(n)[2n]).$$

(b)  $NcMot(X)$ . à la Robalo.

$$K_i(X) \cong \pi_i \Gamma_{\mathbb{P}}(\mathbb{1}, \mathbb{1}).$$

Get normed  $\mathbb{E}_\infty$ -algebras.

Thm.  $H\mathbb{Z}, KGL$  are normed motivic spectra.

So is  $\Gamma GL$ .

Non-ex.  $\mathcal{S}[\mathbb{Z}^{-1}]$  Hopf-motivic spectra.

This is  $\mathbb{E}_\infty$  but not normed. Same with  $\mathcal{S}[\mathbb{p}^{-1}]$ .

Rem. Get normed equivalent  $\mathbb{E}_\infty$ -rings for the étale fundamental groups.