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AKTS.

FM partners of Enriques and bielliptic surfaces
in positive char.

X/k sm. proj. $k=\bar{k}$.

$D(X) := D^b(X)$.

Big Q. What is $FM(X) := \{Y : D(Y) \simeq D(X)\}$.

Sub Q. What happens in char. p specifically.

Use Torelli theorems $/\mathbb{C}$.
Recover from cohomology.

$$H^2(X, \mathbb{Z}) \underset{\text{Hodge}}{\simeq} H^2(X', \mathbb{Z}) \qquad X, X' \text{ K3s}$$
$$\Rightarrow X \cong X'.$$

Thm (Bridgeland – Maciocia). $X/\mathbb{C}$ bielliptic or Enriques: $FM(X) = \{X\}$.

Thm (H.–Lieblich–Tirabassi). Same holds if $k=\bar{k}$ char $k \geq 3$ (Enriques)
or ≥ 5 (bielliptic).

Thm (Orlov). Given $F : D(X) \simeq D(Y)$, then there exists $\mathcal{E} \in D(X \times Y)$ s.t. $F \cong \Phi_{\mathcal{E}} = p_{2*}(p_1^* \otimes \mathcal{E})$.

Ex (Mukai). $D(A) \simeq D(\hat{A})$ w/ kernel the Poincaré bundle. A abelian.

Action descends to Chow groups.

$$\Phi_{\mathcal{E}}^{CH} : CH^*_{num}(X)_{\mathbb{Q}} \to CH^*_{num}(Y)_{\mathbb{Q}}$$

Use Mukai vector

$$v(\mathcal{E}) = ch(\mathcal{E})\sqrt{td(X)}$$

$$\begin{tikzcd} D(X) \arrow[r, "\Phi_{\mathcal{E}}"] \arrow[d, "v(-)"'] & D(Y) \arrow[d, "v(-)"] \\ CH(X) \arrow[r, "\Phi_{\mathcal{E}}^{CH}"] & CH(Y). \end{tikzcd}$$

Enriques and bielliptic surfaces.

X, $\omega_X^{\otimes n} \cong \mathcal{O}_X$ ($n = 2$ X Enriques, $n \in \{2,3,4,6\}$ X bielliptic).

Canonical cover $\tilde{X} := \underline{\mathrm{Spec}}_X \bigoplus_{i=0}^{n-1} \omega_X^{\otimes i} \to X$

X Enriques : $\tilde{X}$ K3.

X bielliptic : $\tilde{X}$ abelian surface.

If E is a sheaf on X s.t. $E \otimes \omega_X \cong E$, then E is descended from $\tilde{X}$.

Bridgeland – Maciocia: $\exists \tilde{\mathcal{E}}$ s.t.

$$\begin{tikzcd} D(\tilde{X}) \arrow[r, "\sim", "\Phi_{\tilde{\mathcal{E}}}"'] \arrow[d, shift right, "\pi_{X*}"'] & D(\tilde{Y}) \arrow[d, shift right, "\pi_{Y*}"'] \\ D(X) \arrow[r, "\sim", "\Phi_{\mathcal{E}}"'] \arrow[u, shift right, "\pi_X^*"'] & D(Y) \arrow[u, shift right, "\pi_Y^*"'] \end{tikzcd}$$

if n is invertible, at least.

Lifting from char. p to char 0.

$$\tilde{X} \xrightarrow[\text{lci}]{p} X$$

X/k char. p

$$p^* L_X \to L_{\tilde{X}} \to L_p$$

Lift to char 0 is to

a DVR with closed fiber X/k.

E.g. $W(k)$.

Remember: there is alg. closed.

$\boxed{\text{Matsusaka – Mumford}}$ Iso of generic fibers $\Rightarrow$ birational iso of special fibers.

Two parts to lifting: ① Deformation. Obstructions vanish.

Grothendieck
② Algebraization: can algebraize formal schemes with "formal" ample linebundles as well as coherent sheaves w/ proper support.

Lieblich: also perfect complexes.

__But__, lifting the kernel is in general obstructed.

Deforming kernels. Via moduli stack of perfect complexes.

X/k

T/k

$$\underline{\mathrm{Perf}}_X(T) = \left\{ \begin{array}{c} \text{perfect cxs on } X \times_k T \\ \text{universally gluable} \\ \text{simple} \end{array} \right\} \subseteq \mathfrak{M}_{D^b(X)}$$

Thm (Lieblich). $\underline{\mathrm{Perf}}_X \xrightarrow[\text{Mori}]{\text{Keel}} \mathrm{Perf}_X$ a gerbe.

coarse space

Thm (Lieblich–Olsson). If $D(X) \xrightarrow{\Phi_{\mathcal{E}}} D(Y)$ is F.F., then

$$\begin{tikzcd}
X \arrow[r, "\mathcal{E}"] \arrow[dr, "\text{open immersion}"'] & \underline{\mathrm{Perf}}_Y \arrow[d] \\
& \mathrm{Perf}_Y .
\end{tikzcd}$$

$T \qquad T'$

This is one lift to W_n at a time.

$$\begin{tikzcd}
X' \arrow[r, dashed] \arrow[d, dotted] & \mathrm{Perf}_{Y'} \arrow[d] \\
X \arrow[r, hook] & \mathrm{Perf}_Y \simeq \mathrm{Perf}_{Y'} \times_{\mathrm{Spec}\, R} \mathrm{Spec}\, k
\end{tikzcd}$$

Lift using smoothness.

Need to check unobstructedness of Brauer class.