

Elden Elmanto.

20 September 2017.

On motivic infinite loop spaces.

Analogue of infinite loop spaces.

Joint w/ Marc Hoyois, Adnan Khan, Vova Sambo, Maria Yukawa.

Def. $\text{MotSpc}_k = \text{Fun}_{\text{Nis, AI}}(\text{Sm}_k^{\text{op}}, \text{Spc})$.

Rem. This def. is inspired by the equation

$$\text{Spc} = \text{Fun}_{\text{Dixant, IR}}(\text{Mfld}^{\text{op}}, \text{Spc}).$$

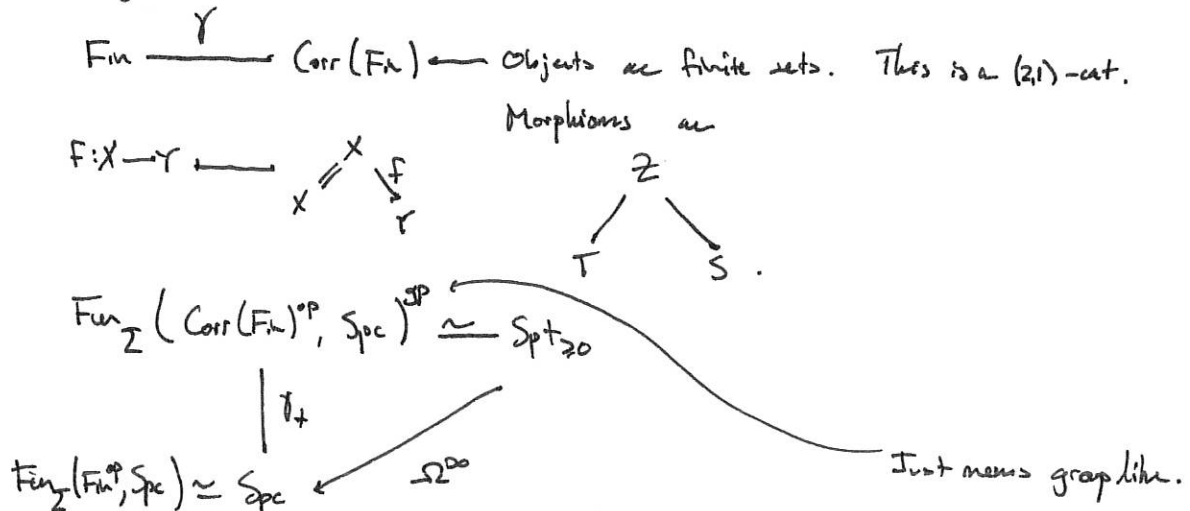
More combinatorially,

$$\text{Spc} = \text{Fun}_{\Sigma}(\text{Fin}^{\text{op}}, \text{Spc})$$

↑
Products preserve products ($\mathbb{1} \rightarrow x$).

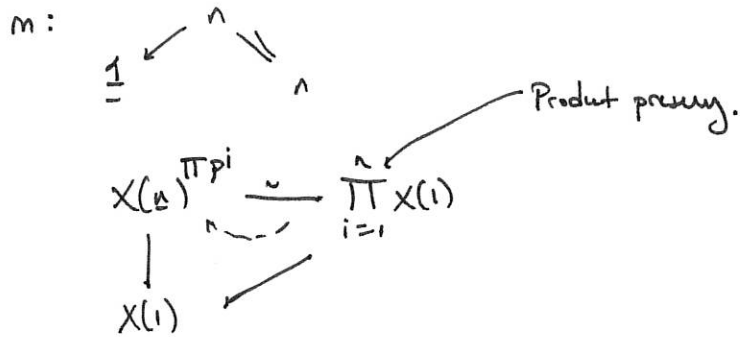
Recall. $\text{Spt} \simeq \text{Spc}_{\neq}^{\wedge}[(S^1)^{\wedge \bullet}]$. Tensor-homot S^1 .

Thm. (Segal, 1974).



Idea: $\Omega^\infty \mathcal{E}$ of a spectrum \mathcal{E} is an infinite loop space. This ~~is~~ is captured by

$$p_i: \underline{1} \longrightarrow \underline{n}$$



in $\text{Corr}(Fin)$.

<u>Topology</u>	<u>Naive AG</u>	<u>Naive HTF</u>
Fin	Sm _S	Sm _S
Corr(Fin)		Corr ^{fr} _k
Spt _{≥0}	Fun _{Nat, A'} (Sm ^{op} _{S, Spt_{≥0})}	(Nat Spt) _{≥0}
Spt	Fun _{Nat, A'} (Sm ^{op} _{S, Spt})	Nat Spt

Goal: fill in the four blanks.

Def. $\text{TotSpt}(k) := \text{TotSpc}(k)_* [(\mathbb{P}^1, *)^{\wedge \infty}]_*$

Remark. $(\mathbb{P}^1, *) \simeq S^1 \wedge G_m$. So, you see also inverting G_m .

Thm (EHKSY). For k an infinite perfect field, there exists a ∞ -cut $\text{Corr}_k^{\text{fr}}$.

$$(\text{TotSpt})_{\geq 0} \simeq \text{Fun}_{\text{Mod } A^1}(\text{Corr}_k^{\text{fr, op}}, \text{Spc})^{\text{gp}}$$

Thm (EHKSY). $\Omega_{\mathbb{P}^1}^{\infty} \mathcal{S}^0$ has the A^1 -homotopy type of an explicit ind-smooth scheme (up to group completion).

$$\text{Use } \text{Hilb}^{\text{Flei}}(A^{\infty}) \overset{\text{open}}{\subseteq} \text{Hilb}(A^{\infty})$$

↑
finite lei subschemes

$$\text{Get } \text{GL}_A \longrightarrow \boxed{X} \longrightarrow \text{Hilb}^{\text{Flei}}(A^{\infty}) \quad \text{GL}_{\infty}\text{-torsor.}$$

$$\text{And, } X^{\text{gp}} \simeq \Omega_{\mathbb{P}^1}^{\infty} \mathcal{S}^0.$$

Thm (EHKSY). For all $n \geq 0$, the motivic Eilenberg-MacLane spaces have models as simplicial ind-smooth schemes.

How to guess the def. of $\text{Corr}_k^{\text{Fr}}$.

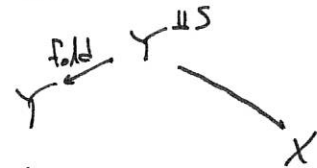
$$\begin{aligned} \text{Fun}_{\text{Nis}, \mathbb{A}^1}(\text{Sm}_k^{\text{op}}, \text{Spt}_{\geq 0}) &\simeq \text{Fun}_{\text{Nis}, \mathbb{A}^1}(\text{Sm}_k^{\text{op}}, \text{Fun}_{\mathbb{Z}}(\text{Corr}(\text{Fin}^{\text{op}}, \text{Spec}))^{\text{gp}}) \\ &\simeq \text{Fun}_{\text{Nis}, \mathbb{A}^1}(\text{Sm}_k^{\text{op}} \times \text{Cor}(\text{Fin}^{\text{op}}, \text{Spec}))^{\text{gp}} \end{aligned}$$

LEMMA of
Hoyois-Bachmann

$$\text{Fun}_{\text{Nis}, \mathbb{A}^1}(\text{Cor}_k^{\text{fold}}, \text{Spec})^{\text{gp}}$$

Objects. $X \in \text{Sm}_k$.

Morphisms.



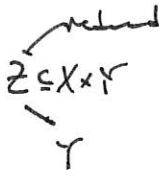
generated by this.

Def (Voevodsky).

SmCor_k

Obj: $X \in \text{Sm}_k$

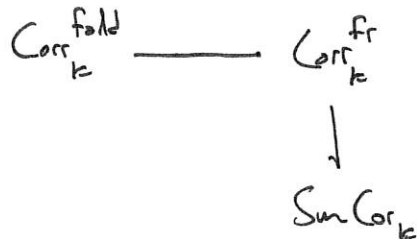
Morphisms: $\mathbb{Z} \left\{ \begin{array}{l} \text{finite} \\ \downarrow \\ i/X \end{array} \right.$



$$\text{Fun}_{\text{Nis}, \mathbb{A}^1}^{\text{add}}(\text{SmCor}_k^{\text{op}}, \text{Mod}_{\mathbb{Z}}) [M(\mathbb{P}^1, \infty)^{-1}] \simeq \text{DM}(k).$$

This is less complex than MotSpt .

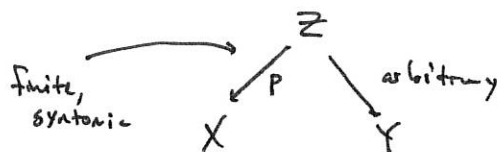
So, we guess our answer sits in



"Def." The ∞ -cat $\text{Corr}_k^{\text{fr}}$.

Obj: $X \in \text{Sm}_k$.

Mps $X \rightarrow Y$ are



together with a map

$$\mathbb{L}_P \rightarrow 0$$

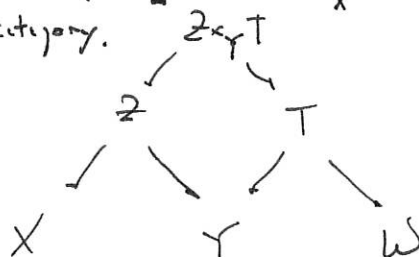
in $\mathcal{L}K(\mathbb{Z})$, the K -theory of \mathbb{Z} .

This is a K -theoretic
triv. of \mathbb{L}_P .

"Stably triv. the normal bundle."

Def. Syntonic = flat + loci.

This is used to compose
in the category.



Looks good.

Lemma. $U \subseteq X^{\text{open}}$ in $S_{n,n}$.
 Svalik - Novobokov.

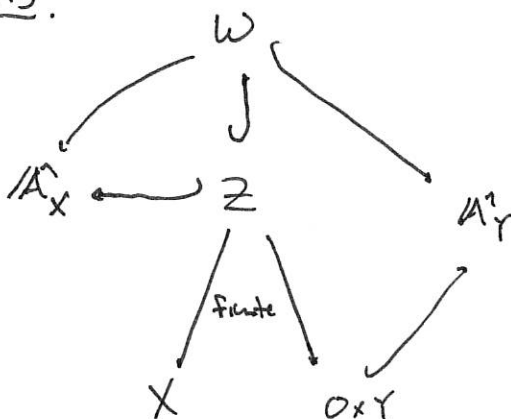
$L_{N \rightarrow X/U}$ in $Sh_{\mathbb{Z}} \rightarrow \mathbb{Z}$ has an explicit pres. as

$$(L_{N \rightarrow X/U})(T) = \begin{cases} Z \subseteq T \text{ closed} \\ \exists! T_h \rightarrow X \\ \psi^{-1}(X/U) \cong Z. \end{cases}$$

$$\text{Hbm } Sh_{\mathbb{Z}} \rightarrow \mathbb{Z} \left(X_+ \wedge (\mathbb{P}^1, \infty)^{\wedge n}, Y_+ \wedge \frac{A^m}{A^n - \{e\}} \right)$$

$X_i, Y_i \in S_{n,n}$ " $\Omega^2 \Sigma^2 \mathbb{Z}$ " $n \rightarrow \infty$ } Gerlender, Anagnostou, Pines, Neshitov.

LHS.



$$\Omega_{\mathbb{P}^1}^{\infty} \Sigma_{\mathbb{P}^1}^{\infty} X_+(X)$$

Framing on \mathbb{Z} .

Normal is framed.



This data is an equatorially framed corr. of level n .

Everything relies on stable triv.