Elden Elmanto.

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On motivic infinite loop spaces.

Analogue of infinite loop spaces.

Joint w/ Marc Hoyois, Adeel Khan, Vova Sosnilo, Maria Yakerson.

Def. $\mathrm{MotSpc}_k = \mathrm{Fun}_{\mathrm{Nis}, \mathbb{A}^1}(\mathrm{Sm}_k^{op}, \mathrm{Spc})$.

Rem. This def. is inspired by the equivalence

$$\mathrm{Spc} \simeq \mathrm{Fun}_{\mathrm{Descent}, \mathbb{R}}(\mathrm{Mfld}^{op}, \mathrm{Spc}).$$

More combinatorially,

$$\mathrm{Spc} \simeq \underset{\substack{\uparrow \\ \text{Functors preserving products } (\sqcup \to \times).}}{\mathrm{Fun}_{\Sigma}}(\mathrm{Fin}^{op}, \mathrm{Spc})$$

Recall. $\mathrm{Spt} \simeq \mathrm{Spc}_*^{\wedge}[(S^1)^{\wedge -1}]$. Tensor-invert S^1.

Thm. (Segal, 1974).

$$\begin{tikzcd} \mathrm{Fin} \arrow[r, "\gamma"] & \mathrm{Corr}(\mathrm{Fin}) \\ f: X \to Y \arrow[r, mapsto] & \begin{tikzcd}[sep=small] & X \arrow[dl, equal] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \end{tikzcd}$$

$\leftarrow$ Objects are finite sets. This is a $(2,1)$-cat. Morphisms are

$$\begin{tikzcd}[sep=small] & Z \arrow[dl] \arrow[dr] & \\ T & & S \end{tikzcd}.$$

$$\begin{tikzcd} \mathrm{Fun}_{\Sigma}(\mathrm{Corr}(\mathrm{Fin})^{op}, \mathrm{Spc})^{gp} \arrow[r, "\simeq"] \arrow[d, "\gamma_*"] & \mathrm{Spt}_{\geq 0} \arrow[dl, "\Omega^{\infty}"] \\ \mathrm{Fun}_{\Sigma}(\mathrm{Fin}^{op}, \mathrm{Spc}) \simeq \mathrm{Spc} & \end{tikzcd}$$

Just means grouplike.

Idea: $\Omega^\infty E$ of a spectrum E is an infinite loop space. This ~~[illegible]~~ is captured by

$$p^i : \underline{1} \longrightarrow \underline{n}$$

$$m: \quad \underline{1} \leftarrow n \;=\!=\; n$$

Product preserving.

$$\begin{tikzcd}
X(\underline{n}) \arrow[r, "\prod p^i", "\sim"'] \arrow[d] & \prod_{i=1}^{n} X(1) \arrow[dl] \\
X(1) &
\end{tikzcd}$$

in $Corr(Fin)$.

Topology	Naive AG	Motivic HTY	
Fin	Sm_S	Sm_S	
$Corr(Fin)$			$Corr_k^{fr}$
$Spt_{\geq 0}$	$Fun_{Nis,\mathbb{A}^1}(Sm_S^{op}, Spt)_{\geq 0}$		$(MotSpt)_{\geq 0}$
Spt	$Fun_{Nis,\mathbb{A}^1}(Sm_S^{op}, Spt)$		$MotSpt$

Goal: fill in the four blanks.

Def. $\mathrm{MotSpt}(k) := \mathrm{MotSpc}(k)_*^{\wedge}\left[(\mathbb{P}^1,*)^{\wedge -1}\right]$.

Remark. $(\mathbb{P}^1,*) \simeq S^1 \wedge \mathbb{G}_m$. So, you are also inverting $\mathbb{G}_m$.

Thm (EHKSY). For k an infinite perfect field, there exists an ∞-cat Corr_k^{fr}.

$$(\mathrm{MotSpt})_{\geq 0} \simeq \mathrm{Fun}_{\mathrm{Nis}, \mathbb{A}^1}\left(\mathrm{Corr}_k^{fr,op}, \mathrm{Spc}\right)^{gp}.$$

Thm (EHKSY). $\Omega^\infty_{\mathbb{P}^1} \mathbb{S}^0$ has the $\mathbb{A}^1$-homotopy type of an explicit ind-smooth scheme (up to group completion).

Use $\underset{\substack{\uparrow \\ \text{finite lci subschemes}}}{\mathrm{Hilb}^{flci}(\mathbb{A}^\infty)} \overset{\text{open}}{\subseteq} \mathrm{Hilb}(\mathbb{A}^\infty)$.

Get $\quad GL_\infty \longrightarrow \overset{X}{\text{[redacted]}} \longrightarrow \mathrm{Hilb}^{flci}(\mathbb{A}^\infty) \quad$ GL_∞-torsor.

And, $X^{gp} \simeq \Omega^\infty_{\mathbb{P}^1} \mathbb{S}^0$.

Thm (EHKSY). For all $n \geq 0$, the motivic Eilenberg–MacLane spaces have models as simplicial ind-smooth schemes.

How to guess the def. of Corr_k^{fr}.

$$\mathrm{Fun}_{Nis,\mathbb{A}^1}(\mathrm{Sm}_k^{op}, \mathrm{Spt}_{\geq 0}) \simeq \mathrm{Fun}_{Nis,\mathbb{A}^1}\left(\mathrm{Sm}_k^{op}, \mathrm{Fun}_\Sigma\left(\mathrm{Corr}(\mathrm{Fin})^{op}, \mathrm{Spc}\right)\right)^{gp}$$

$$\simeq \mathrm{Fun}_{Nis,\mathbb{A}^1}\left(\mathrm{Sm}_k^{op} \times \mathrm{Cor}(\mathrm{Fin})^{op}, \mathrm{Spc}\right)^{gp}$$

$$\underset{\substack{\text{LEMMA of} \\ \text{Hoyois–Bachmann}}}{\simeq} \mathrm{Fun}_{Nis,\mathbb{A}^1}\left(\mathrm{Cor}_k^{fold}, \mathrm{Spc}\right)^{gp}.$$

Objects. $X \in \mathrm{Sm}_k$.

Morphisms.
$$\begin{tikzcd} & Y^{\amalg S} \arrow[dl, "fold"'] \arrow[dr] & \\ Y & & X \end{tikzcd}$$

generated by them.

Def (Voevodsky).

SmCor_k

Obj: $X \in \mathrm{Sm}_k$

Morphisms: $\mathbb{Z}\left\{ \begin{tikzcd} & Z \subseteq X \times Y \arrow[dl, "finite"'] \arrow[dr] & \\ X & & Y \end{tikzcd} \right.$ reduced, surjective on components

$$\mathrm{Fun}^{add}_{Nis,\mathbb{A}^1}\left(\mathrm{SmCor}_k^{op}, \mathrm{Mod}_\mathbb{Z}\right)\left[M(\mathbb{P}^1,\infty)^{-1}\right] \simeq \mathrm{DM}(k).$$

This is less complex than MotSpt.

So, we guess our answer sits in

$$\begin{tikzcd} \mathrm{Corr}_k^{\mathrm{fold}} \arrow[r, dash] & \mathrm{Corr}_k^{\mathrm{fr}} \arrow[d] \\ & \mathrm{SmCor}_k \end{tikzcd}$$

"Def." The ∞-cat $\mathrm{Corr}_k^{\mathrm{fr}}$.

Obj: $X \in \mathrm{Sm}_k$.

Maps $X \to Y$ are

$$\begin{tikzcd} & Z \arrow[dl, "p"'] \arrow[dr, "\text{arbitrary}"] & \\ X & & Y \end{tikzcd}$$

finite, syntomic

together with a map

$$\mathbb{L}_p \longrightarrow 0$$

in $\Omega^\infty K(Z)$, the K-theory of Z.

This is a K-theoretic triv. of $\mathbb{L}_p$.

"Stably triv. the normal bundle."

__Def.__ Syntomic = flat + lci.

Zariski loc

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] & V \arrow[dl, "\text{r.b.}"] \\ X & \end{tikzcd}$$

regular

This is used to compose in the category.

$$\begin{tikzcd} & & Z\times_Y T \arrow[dl] \arrow[dr] & & \\ & Z \arrow[dl] \arrow[dr] & & T \arrow[dl] \arrow[dr] & \\ X & & Y & & W \end{tikzcd}$$

Looks good.

Lemma. $U \subseteq X$ open in Sm_k.

Suslin–Voevodsky.

$L_{Nis} X/U$ in Shv_{Nis} has an explicit pres. as

$$\left(L_{Nis} X/U\right)(T) = \left\{ \begin{array}{l} Z \subseteq T \text{ closed} \\ \phi : T_Z^h \longrightarrow X \\ \phi^{-1}(X/U) \cong Z. \end{array} \right.$$

$$\mathrm{Hom}_{Shv_{Nis,\bullet}}\left(X_+ \wedge (\mathbb{P}^1,\infty)^{\wedge n},\ Y_+ \wedge \frac{\mathbb{A}^n}{\mathbb{A}^n - \{0\}} \right) \qquad \frac{Y\times\mathbb{A}^n}{Y\times\mathbb{A}^n - Y\times\{0\}}$$

$X, Y \in Sm_k$ "$\Omega^n \Sigma^n Y$"

$n \longrightarrow \infty$ ‖ Garkusha, Ananyevskiy, Panin, Neshitov.

$$\Omega^\infty_{\mathbb{P}^1} \Sigma^\infty_{\mathbb{P}^1} Y_+(X)$$

LHS.

$$\begin{tikzcd}
 & W \arrow[dl, bend right] \arrow[d, hook] \arrow[drr, bend left] & & \\
\mathbb{A}^n_X & Z \arrow[l, hook'] \arrow[dl, "\text{finite}"'] \arrow[d] & & \mathbb{A}^n_Y \\
X & O \times Y \arrow[urr, bend right] & &
\end{tikzcd}$$

Framing on Z.

Normal is trivial.

This data is an equationally framed corr. of level n.

Everything relies on stable triv.