MATH 215 – Practice Midterm

We will assume the existence of a set $\mathbb{R}$, whose elements are called **real numbers**, along with a well-defined binary operation $+$ on $\mathbb{R}$ (called addition), a second well-defined binary operation $\cdot$ on $\mathbb{R}$ (called multiplication), and a relation $<$ on $\mathbb{R}$ (called less than), and that the following fourteen statements involving $\mathbb{R}$, $+$, $\cdot$, and $<$ are true:

A1. For all a, b, c in $\mathbb{R}$, $(a + b) + c = a + (b + c)$.
A2. There exists a unique real number 0 in $\mathbb{R}$ such that $a + 0 = 0 + a = a$ for every real number a.
A3. For every a in $\mathbb{R}$, there exists a unique real number $-a$ in $\mathbb{R}$ such that $a + (-a) = (-a) + a = 0$.
A4. For all a, b in $\mathbb{R}$, $a + b = b + a$.

M1. For all a, b, c in $\mathbb{R}$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
M2. There exists a unique real number 1 in $\mathbb{R}$ such that $a \cdot 1 = 1 \cdot a = a$ for all a in $\mathbb{R}$.
M3. For all non-zero a in $\mathbb{R}$, there exists a unique real number a^{-1} in $\mathbb{R}$ such that $a \cdot a^{-1} = a^{-1} \cdot a = 1$.
M4. For all a, b in $\mathbb{R}$, $a \cdot b = b \cdot a$.

D1. For all a, b, c in $\mathbb{R}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.
NT1. $1 \neq 0$.
O1. For all a in $\mathbb{R}$, exactly one of the following statements is true: $0 < a$, $a = 0$, $0 < -a$.
O2. For all a, b in $\mathbb{R}$, if $0 < a$ and $0 < b$, then $0 < a + b$.
O3. For all a, b in $\mathbb{R}$, if $0 < a$ and $0 < b$, then $0 < a \cdot b$.

We also assume the existence of sets of **natural numbers** $\mathbb{N} = \{1, 2, 3, \ldots\}$ and of **integers** $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ with $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{R}$. We assume the following basic properties: (i) if a, b are integers, then $a + b$, $a - b$, and ab are integers; (ii) if a, b are natural numbers, then $a + b$ and ab are natural numbers.

Definition 1 *A* **rational** *number is a real number x such that there exists a natural number q such that $q \cdot x$ is an integer. A real number is* **irrational** *if it is not rational.*

Proposition 2 *Prove that if x is irrational and y is rational and non-zero, then $x \cdot y$ is irrational.*

Proposition 3 *If x is irrational and y is rational, then $x + y$ is irrational.*

Proposition 4 *If x is an irrational number, then there exists a unique real number x^{-1} such that $x \cdot x^{-1} = x^{-1} \cdot x = 1$.*

Proposition 5 *If x is an irrational number, then x^{-1} is irrational.*

Proposition 6 *Let w, x, y be real numbers. If $x \cdot y = x \cdot z$ and $x \neq 0$, then $y = z$.*