MATH 215 – Practice Final

We will assume the existence of a set $\mathbb{Z}$, whose elements are called integers, along with a well-defined binary operation $+$ on $\mathbb{Z}$ (called addition), a second well-defined binary operation $\cdot$ on $\mathbb{Z}$ (called multiplication), and a relation $<$ on $\mathbb{Z}$ (called less than), and that the following fourteen statements involving $\mathbb{Z}$, $+$, $\cdot$, and $<$ are true:

A1. For all a, b, c in $\mathbb{Z}$, $(a + b) + c = a + (b + c)$.
A2. There exists a unique integer 0 in $\mathbb{Z}$ such that $a + 0 = 0 + a = a$ for every integer a.
A3. For every a in $\mathbb{Z}$, there exists a unique integer $-a$ in $\mathbb{Z}$ such that $a + (-a) = (-a) + a = 0$.
A4. For all a, b in $\mathbb{Z}$, $a + b = b + a$.
M1. For all a, b, c in $\mathbb{Z}$, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
M2. There exists a unique integer 1 in $\mathbb{Z}$ such that $a \cdot 1 = 1 \cdot a = a$ for all a in $\mathbb{Z}$.
M4. For all a, b in $\mathbb{Z}$, $a \cdot b = b \cdot a$.
D1. For all a, b, c in $\mathbb{Z}$, $a \cdot (b + c) = a \cdot b + a \cdot c$.
NT1. $1 \neq 0$.
O1. For all a in $\mathbb{Z}$, exactly one of the following statements is true: $0 < a$, $a = 0$, $0 < -a$.
O2. For all a, b in $\mathbb{Z}$, if $0 < a$ and $0 < b$, then $0 < a + b$.
O3. For all a, b in $\mathbb{Z}$, if $0 < a$ and $0 < b$, then $0 < a \cdot b$.
O4. For all a, b in $\mathbb{Z}$, $a < b$ if and only if $0 < b + (-a)$.
WOP. If S is a non-empty set of non-negative integers, then S has a least element.

Remark 1 *The above axiom is referred to as the* **Well-Ordering Principle (WOP)**. *We will assume it is true without proof.*

Proposition 2 (20 points) *(a) Let a be an integer and n a natural number. State the division algorithm for a and n.*

(b) Let a and b be integers. Define what it means for a to divide b.

(c) Let a, b be integers and let n be a natural numbers. Define $a \equiv b \bmod n$.

(d) Let S be a set of integers. Define what it means for $\ell \in S$ to be a least element.

Proposition 3 (10 points) *Let a, b, c be integers, and let n be a natural number. Prove that if $a \equiv b \bmod n$, then $ac \equiv bc \bmod n$.*

Theorem 4 (10 points) *Let $P(k)$ denote a statement for every integer $k = 0, 1, 2, \ldots$. If the following are true:*

1. *$P(0)$ is true; and*
2. *The truth of $P(\ell - 1)$ implies the truth of $P(\ell)$ for every integer $\ell = 1, 2, 3, \ldots$,*

then $P(k)$ is true for all integers $k = 0, 1, 2, 3 \ldots$.

Problem 5 (10 points) *Find the greatest common divisor of* 270 *and* 192*. Then, find integers* m *and* n *such that* $\gcd(270, 192) = 270m + 192n$*.*

Proposition 6 (10 points) *Let* a *be an integer and* n *a natural number. Show that there exists an integer* r *such that* $a \equiv r \bmod n$ *and* $0 \leq r < n$*.* **Note:** *you do not need to prove uniqueness.*

Proposition 7 (10 points) *Prove that for all* $n \geq 0$*,*

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

Proposition 8 (10 points) *Prove that* $a^2 - 1$ *is divisible by* 8 *for all* **odd** *integers* a*.*

Proposition 9 (10 points) *Prove that for all* $1 \leq k \leq n$ *one has*

$$\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}.$$