Rational Homotopy Theory - Lecture 20

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1. Brief return to minimal cdgas

Recall minimal cdgas.

Proposition 1.1. *If X is a minimal cdga, then X is cofibrant, and each $X(n,q) \to X(n,q+1)$ is a cofibration.*

Proof. In fact, it suffices to check that each $X(n,q) \to X(n,q+1)$ is a cofibration. Indeed, this is an elementary extension, say given as $X(n,q+1) = X(n,q) \otimes_d \Lambda_n(V)$, where V is a $\mathbb{Q}$-vector space. We can write $X(n,q) \to X(n,q+1)$ as a pushout

$$\begin{array}{ccc} \bigotimes_\alpha S(n+1)_\alpha & \longrightarrow & X(n,q) \\ \downarrow & & \downarrow \\ \bigotimes_\alpha D(n+1)_\alpha & \longrightarrow & X(n,q+1), \end{array}$$

indexed over a basis $\{\alpha\}$ of V. Since pushouts of cofibrations are cofibrations, the result follows. $\square$

Proposition 1.2. *If $f : X \to Y$ is a weak equivalence of minimal cdgas, then f is an isomorphism.*

Proof. By what we have seen above, $\pi(f) : \pi X \cong \pi Y$. Since the differential of a minimal cdga is decomposable, $\pi X \cong \mathrm{Q}X$ and $\pi Y \cong \mathrm{Q}Y$. But, X is a free graded-commutative algebra on $\mathrm{Q}X$, and similarly for Y. $\square$

We saw long ago that if X satisfies $\mathrm{H}^0 A \cong \mathbb{Q}$ (i.e., if X is coconnected), then there is a minimal cdga M and a quasi-isomorphism $M \to X$. In fact, M is unique up to non-unique isomorphism.

Proposition 1.3. *If $M \to X$ and $N \to X$ are weak equivalences of cdgas with M and N minmal, then $M \cong N$.*

Proof. This follows from the fact that $[M,N] \to [M,X]$ is a bijection for M cofibrant. $\square$

Theorem 1.4 (Whitehead)**.** *Let $f : W \to X$ be a map of strictly coconnected cofibrant cdgas, and let $n \geq 1$. The induced map $f^* : \mathrm{H}^i W \to \mathrm{H}^i X$ is an isomorphism for $i \leq n$ and an injection for $i = n+1$ if and only if $f^* : \pi^i W \to \pi^i X$ is an isomorphism for $i \leq n$ and an injection for $i = n+1$.*

Proof. Suppose first that we have an isomorphism (injection) on cohomology in the stated range. Produce a minimal factoriazation, as in the construction of a minimal model. Namely, factor f as $W \xrightarrow{i} M_f \xrightarrow{p} X$ in pointed cdgas, where i is a cofibration and p is a quasi-isomorphism. We can assume that this is done in such away that i is a sequence of elementary extensions. Consider the long exact sequence in homotopy groups associated to $W \to M_f \to M_f/W$. Since i is minimal, the first new generator is in degree $n+1$, so that $\mathrm{Q}i$ is an isomorphism up to degree n and an injection in degree $n+1$. This proves that $\pi^i(M_f/W) = 0$ for $1 \leq i \leq n$, as desired.

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Now, suppose that we have an isomorphism (injection) in homotopy in the stated range. Let $M_W \xrightarrow{g} M_X$ be minimal algebras resolving W and X. That is, $M_W \to W$ and $M_X \to X$ are minimal models, and there is a commutative diagram

$$\begin{array}{ccc} M_W & \longrightarrow & M_X \\ \downarrow & & \downarrow \\ W & \longrightarrow & X. \end{array}$$

Then, g also induces an isomorphism (injection) in homotopy in the stated range. Since M_W and M_X are minimal, this means that in fact as cdgas they are isomorphic up to degree n, and g induces an injection in degree n. □

Exercise 1.5. Show that any cofibrant cdga is a tensor product of a minimal cdga and possibly infinitely many copies of $D(n)$ as n varies.

References

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