Rational Homotopy Theory - Lecture 16

BENJAMIN ANTIEAU

Basically, we discuss the same material in lecture on 10 March 2016 as well.

1. THE PL DE RHAM THEOREM

We are going to take a slightly different approach, based on the presentation in Félix-Halperin-Thomas [3], with some category-theoretical improvements to make our lives easier.

Recall that we have the simplicial cdga \( V(\bullet, \ast) \), and the rational PL de Rham complex of a simplicial set \( X \) is

\[
A^*(X) = \text{Hom}_{sSets}(X, V(\bullet, \ast)).
\]

Now, given any simplicial dga \( R(\bullet, \ast) \), we let

\[
A^*_R(X) = \text{Hom}_{sSets}(X, R(\bullet, \ast)).
\]

So, as an example, we have \( A^*(X) = A^*_V(X) \). We call \( A^*_R(X) \) the \textbf{cochains on X with coefficients in R}.

We will introduce a simplicial dga \( N \) such that \( A^*_N \) is naturally isomorphic to \( \text{N}^*(X) \), the normalized cochain algebra of \( X \). In fact, let

\[
\text{N}(\bullet, \ast) = \text{N}^q(\Delta^\ast).
\]

**Lemma 1.1.** For any simplicial set \( X \), the natural map \( \text{A}^*_N(X) \to \text{N}^*(X) \) is an isomorphism.

**Proof.** Let \( f \in A^q_N(X) = \text{Hom}_{sSets}(X, N(\bullet, q)) \). For a \( p \)-simplex \( \tau \) of \( X \), let \( f_\tau \in N(p, q) \), be the normalized \( q \)-cochain on \( \Delta^p \). Given a \( q \)-simplex \( \sigma \in X_q \), we can apply \( f \) to obtain \( f_\sigma = f(\sigma)e_q \in N(q, q) = Q \cdot e_q \), where \( e_q \) is dual to the fundamental simplex \( e_q \) of \( \Delta^q \). One checks that \( \sigma \mapsto f(\sigma) \) defines an element of \( N^q(X) \), and that the assignment \( \text{A}^*_N(X) \to \text{N}^*(X) \) is a dga map. If \( f \) vanishes on all \( q \)-simplices, then it must vanish on all simplices of \( X \). To see this, let \( \tau : \Delta^p \to X \) be a \( p \)-simplex of \( X \), and let \( \alpha : \Delta^q \to \Delta^p \) be some composition of face and degeneracy maps. Since \( f \) is a simplicial map, \( f_\tau(\alpha) = f_\circ(\epsilon_q) = 0 \).

Now, suppose that \( F \in \text{Hom}(X_q, Q) \) is a normalized cochain, so that \( F(\sigma_i(\tau)) = 0 \) for any \( i \) and \( \tau \in X_{q-1} \). Let \( \tau : \Delta^p \to X \), and define \( f(\tau) = N^p(F) \in N^q(\Delta^p) = N(p, q) \). Hence, \( \text{A}^*_N(X) \to \text{N}^*(X) \) is surjective. \( \square \)

**Theorem 1.2.** The natural maps

\[
\text{A}^*_N(X) \to \text{A}^*_N \otimes \text{V}(X) \leftarrow \text{A}^*_V(X)
\]

are quasi-isomorphisms of dgas for any simplicial set \( X \).

We will need some more preliminaries before proving this. We call a simplicial dga \( R(\bullet, \ast) \) \textbf{degree-wise contractible} if \( R(\bullet, q) \) the simplicial abelian group is contractible for all \( q \). Note that in Félix-Halperin-Thomas this property is called ‘extendable’. But, we will just call it what it is.

**Proposition 1.3.** Let \( G_\bullet \) be a simplicial group. Then, \( G_\bullet \) is fibrant as a simplicial set.

**Proof.** Recall that in order to be fibrant, dotted lifts must exist in any solid-arrow diagram

\[
\begin{array}{ccc}
A^q_k & \rightarrow & G \\
\downarrow & & \\
\Delta^q & \rightarrow & \ast.
\end{array}
\]

\textbf{Date:} 8 March 2016.
This is equivalent to the following condition: for any \(x_0, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n \in G_{n-1}\) such that \(\partial_i x_j = \partial_{j-1} x_i\), \(i < j\) and \(i, j \neq k\), there exists \(y \in G_n\) such that \(\partial_i y = x_i\) for \(i \neq k\). We construct a filling \(y\) inductively as follows. Let \(g_{-1} = 1\), the identity element of \(G_n\). Assume we have constructed \(g_{r-1}\) such that \(\partial_i g_{r-1} = x_i\) for \(0 \leq i \leq r-1\), \(i \neq k\). If \(r = k\), set \(g_r = g_{r-1}\). Otherwise, if \(r \neq k\), define \(u = x_r^{-1} \partial_r (g_{r-1})\). If \(i < r\),
\[
\partial_i (u) = \partial_i (x_r^{-1}) \partial_r g_{r-1} = (\partial_i x_r)^{-1} \partial_r g_{r-1} = (\partial_i x_r)^{-1} \partial_r x_i = 1,
\]
by hypothesis on the \(x_i\). Thus, if we set \(g_r = g_{r-1}(\sigma_r u)^{-1}\), we have \(\partial_i (g_r) = x_i (\sigma_{r-1} \partial_i (u))^{-1} = x_i\) if \(i < r\), and \(\partial_r (g_r) = \partial_r (g_{r-1}) u^{-1} = x_r\). Thus, taking \(y = g_n\) works. 

**Remark 1.4.** Xing Gu asked in class why this proof does not work to show that \(G_*\) satisfies the lifting property with respect to all diagrams

\[
\begin{array}{ccc}
\partial \Delta^n & \to & G \\
\downarrow & & \downarrow \\
\Delta^n & \to & *
\end{array}
\]

In other words, why doesn’t the proof show moreover that \(G_*\) is contractible. The basic reason is as follows. If we took a sequence \(x_0, \ldots, x_n \in G_n\) such that \(\partial_i x_j = \partial_{j-1} x_i\) as in the proof, then the proof would work to construct \(g_{n-1}\) such that \(\partial_i (g_{n-1}) = x_i\) for \(0 \leq i \leq n-1\). What happens in degree \(n\)? We define \(u = x_n^{-1} \partial_n (g_{n-1})\), and then we set \(g_n = g_{n-1}(\sigma_n u)^{-1}\).

All good, right? **Wrong!** The class \(u\) is an \(n-1\)-simplex, so there is no \(n\)th degeneracy map to apply to it! This is related to the fact that a connected simplicial set with an **extra degeneracy** is contractible. If we had an extra degeneracy, the proof would work.

Here are a couple remarks related to this question. Recall that if \(G\) is a group, \(BG\) is the simplicial set with \(BG_n = G^n\) (so that \(BG_0 = \ast\)). The face maps are given by \(\sigma_i (g_1, \ldots, g_n) = (g_1, \ldots, \hat{g_i}, \ldots, g_n)\) and
\[
\partial_i (g_1, \ldots, g_n) = \begin{cases} 
(g_2, \ldots, g_n) & \text{if } i = 0, \\
(g_1, \ldots, g_i g_{i+1}, g_{i+2}, \ldots, g_n) & \text{if } 0 < i < n, \\
(g_1, \ldots, g_{n-1}) & \text{if } i = n.
\end{cases}
\]

As mentioned before I think, \(BG\) is called the **classifying space** of \(G\), and indeed we have \(|BG|\) is a \(K(G,1)\)-space.

**Exercise 1.5.** Show that \(BG\) is a simplicial group if and only if \(G\) is abelian.

**Exercise 1.6.** Let \(A\) be an abelian group. Prove that by hand that if every diagram

\[
\begin{array}{ccc}
\partial \Delta^2 & \to & BA \\
\downarrow & & \downarrow \\
\Delta^2 & \to & *
\end{array}
\]

has a lift, then \(A = 0\).

**Lemma 1.7.** Suppose that \(R(\bullet, \ast)\) is degree-wise contractible and that \(X \subseteq Y\) is an inclusion of simplicial sets. Then, \(\Lambda^\ast_X(Y) \to \Lambda^\ast_X(X)\) is surjective.

**Proof.** Since \(R(\bullet, q)\) is a Kan complex for all \(q\), contractibility implies that \(R(\bullet, q) \to \ast\) is an acyclic fibration. But, \(X \to Y\) is a cofibration. It follows that there is always a lift in the
diagram
\[
\begin{array}{ccc}
X & \longrightarrow & R(\bullet, q) \\
\downarrow & & \downarrow \\
Y & \longrightarrow & *_{s}
\end{array}
\]
for any \( q \). This proves the lemma. \qed

**Example 1.8.** We saw that \( \nabla(\bullet, \ast) \) is degree-wise contractible in Lecture 14.

**Lemma 1.9.** The simplicial dga \( N \) is degree-wise contractible.

By construction, \( H^*(N(p, \ast)) \) is the cellular Q-cohomology of \( \Delta^p_{op} \), which is a contractible space, so it vanishes in positive degrees and is \( Q \) in degree 0.

**Proof.** Consider \( N(\bullet, q) \). As in the argument for the contractibility of \( \nabla(\bullet, q) \), it is enough to consider the \( q = 0 \) case, since it is enough to show that the homology of \( N(\bullet, q) \) vanishes, and this is a graded module over the graded ring \( N(\bullet, 0) \). Now, consider \( N(1, 0) \Rightarrow N(0, 0) \). Note that \( N(p, 0) = \text{Hom}(\Delta^p, Z) \cong \mathbb{Q}^{p+1} \). With the natural basis, \( N(1, 0) \Rightarrow N(0, 0) \) is \( \mathbb{Q}^2 \Rightarrow \mathbb{Q} \). The chain complex associated to \( N(\bullet, 0) \) has lowest differential \( \partial_0 - \partial_1 : \mathbb{Q}^2 \rightarrow \mathbb{Q} \), which we can write in matrix form as \((-1 \quad 1\)). Evidently this is surjective, so that there is no degree zero homology. Since \( H_0N(\bullet, 0) \) has a ring structure via the Alexander-Whitney map, and since \( 1 = 0 \) in this ring, we have that the ring is zero, as desired. \( \square \)

Given a pair \( Y \subseteq X \) and a degree-wise contractible simplicial dga \( R \), we define \( A^*_R(X, Y) \) to be the kernel of \( A^*_R(X) \rightarrow A^*_R(Y) \). These are the **cochains of the pair with coefficients in** \( R \).

**Proposition 1.10.** If \( R \rightarrow S \) is a map of degree-wise contractible simplicial dgas such that \( R(p, \ast) \rightarrow S(p, \ast) \) is a quasi-isomorphism for all \( p \geq 0 \), then \( A^*_R(X, Y) \rightarrow A^*_S(X, Y) \) is a quasi-isomorphism for all pairs \( Y \subseteq X \).

**Proof.** It is enough to prove the proposition for \( Y = \emptyset \), so that we just have to prove that \( A^*_R(X) \rightarrow A^*_S(X) \) is a quasi-isomorphism for all simplicial sets \( X \). Note that \( A^*_R(\Delta^p) \cong R(p, \ast) \) and \( A^*_S(\Delta^p) \cong S(p, \ast) \), by representability. Let \( s_kX \) be the \( n \)-skeleton of \( X \). Note that \( s_kX = \Delta^k \), the disjoint union of the \( 0 \)-simplices of \( X \). Since this is a coproduct,

\[
\prod_{\Delta^0 \rightarrow X} \Delta^0,
\]

it follows from our hypothesis that \( A^*_R(s_kX) \rightarrow A^*_S(s_kX) \) is a quasi-isomorphism. We prove by induction that if the claim is true for all \( p - 1 \)-dimensional simplicial sets, then it is true for all \( n \)-dimensional simplicial sets. So, assume that \( p - 1 \geq 0 \) and that \( A^*_R(s_{k-1}X) \rightarrow A^*_S(s_{k-1}X) \) is a quasi-isomorphism for all simplicial sets \( X \). Note that this includes the boundary \( \partial \Delta^p \). Since we know that we get a quasi-isomorphism for \( \Delta^p \), this implies that all three vertical maps are quasi-isomorphisms in

\[
\begin{array}{cccc}
0 & \longrightarrow & A^*_R(\Delta^p, \partial \Delta^p) & \longrightarrow & A^*_R(\Delta^p) & \longrightarrow & A^*_R(\partial \Delta^p) & \longrightarrow & 0 \\
0 & \longrightarrow & A^*_S(\Delta^p, \partial \Delta^p) & \longrightarrow & A^*_S(\Delta^p) & \longrightarrow & A^*_S(\partial \Delta^p) & \longrightarrow & 0.
\end{array}
\]

Suppose that \( Y \) is \( p - 1 \)-dimensional, and that \( X \) is obtained from \( Y \) by adding a single non-degenerate \( p \)-simplex \( \sigma \). Note that in this case, the boundary of \( \sigma \) is contained in \( Y \). In this case, \( A^*_R(X, Y) \cong A^*_R(\Delta^p, \partial \Delta^p) \), and similarly for \( S \). Indeed, both sides are completely determined by where they send the unique \( p \)-simplex not in \( Y \) or \( \partial \Delta^p \), respectively. It follows that \( A^*_R(s_{k-1}X) \rightarrow A^*_S(s_{k-1}X) \) is a (possibly transfinite) filtered limit of quasi-isomorphisms, and hence it is a quasi-isomorphism by the lemma below when \( I \) is sufficiently small. Since \( X = \text{colim}_p s_kX \), we again have \( A^*_R(X) = \text{lim}_p A^*_R(s_{k-1}X) \), the next lemma works for \( X \) since \( \mathbb{N} \) is \( \mathbb{N}_1 \)-small. In the general case for going from \( s_{k-1}X \) to \( s_kX \), it is better to argue
that $A^*_R(skp X, skp_{−1} X) \cong \bigoplus \Delta^p(\partial \Delta^p)$ where the direct sum is over all non-degenerate $p$-simplices of $X$.

**Lemma 1.11.** Suppose that $I$ is an $\aleph_\omega$-small filtered category, and let $F, G : I^{op} \to \text{Ch}^{>0}$ be functors from $I^{op}$ to non-negatively graded cochain complexes with a natural transformation $F \to G$. If $F(i) \to G(i)$ is a quasi-isomorphism for all $i \in I$, then $\lim_{i^{op}} F(i) \to \lim_{i^{op}} G(i)$ is a quasi-isomorphism.

**Proof.** Since $I$ is small and filtered, the derived functors $R^p \lim$ vanish for $p >> 0$ by work of Jensen (1970). It follows that the spectral sequence $E^{p,q}_2 = R^p \lim_i H^q(F(i)) \Rightarrow H^{p+q}(\lim_i F(i))$ converges, from which the lemma follows from the functoriality of spectral sequences. □

**Question 1.12.** Can we prove the lemma in full generality for small filtered $I$ using homotopy limits and model categories?

**Proof of Theorem 1.2.** We can apply Proposition 1.10 to the two morphisms $N \to N \otimes \nabla \leftarrow \nabla$. We only have to observe that $N \otimes \nabla$ is degree-wise contractible. In degree $q$, we have

$$(N \otimes \nabla)(\bullet, q) \cong \bigoplus_{a+b=q} N(\bullet, a) \otimes \nabla(\bullet, b).$$

The homology of each summand on the right side vanishes by Küneth. □

What’s very nice about this approach is that we get multiplicativity without further work, and this answers Thom’s question completely.

**References**


