Rational Homotopy Theory - Lecture 12

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1. Geometric realization of simplicial sets

Let $X_\bullet$ be a simplicial set. There is a space $|X_\bullet|$ naturally associated to $X_\bullet$ called the **geometric realization** of X. It is given as follows. First, there is a high-brow way of defining it. Let

$$|X| = \operatorname*{colim}_{\Delta^n \to X} \Delta^n_{\mathrm{top}}.$$

This is a Kan extension. Indeed, let $\mathrm{Simplex} \subseteq \mathrm{sSets}$ be the full subcategory consisting of the objects Δ^n for $n \geq 0$. Let $|-| : \mathrm{Simplex} \to \mathrm{Top}$ be the natural functor that takes Δ^n to Δ^n_{top}. (The category $\mathrm{sSets}_{/X}$ is called the **simplex category** of X.) Then,

$$|-| : \mathrm{sSets} \to \mathrm{Top}$$

is the left Kan extension, making the following diagram commute:

$$\begin{array}{ccc} \mathrm{Simplex} & \xrightarrow{|-|} & \mathrm{Top} \\ \downarrow & \nearrow_{|-|} & \\ \mathrm{sSets} & & \end{array}.$$

A more down-to-earth description of the geometric realization is as a quotient

$$|X| = \left(\coprod_{n \geq 0} X_n \times \Delta^n_{\mathrm{top}} \right) / \sim,$$

where $(s, f(x)) \sim (f^*(s), x)$ for any map $f : [m] \to [n]$, any point $x \in \Delta^m_{\mathrm{top}}$ and any simplex $s \in X_n$.

Exercise 1.1. Recall that a simplex $s \in X_n$ is **degenerate** if $s = \sigma_i(t)$ for some $t \in X_{n-1}$ and some i. Let $X_n^{\mathrm{ess}} \subseteq X_n$ be the set of non-degenerate n-simplices. Show that the natural map

$$\left(\coprod_{n \geq 0} X_n^{\mathrm{ess}} \times \Delta^n_{\mathrm{top}} \right) / \sim' \to |X|$$

is a weak homotopy equivalence, where $\sim'$ is the restriction of $\sim$ to the non-degenerate simplices.

In any case, we have the following crucial result.

Proposition 1.2. *The functors $|-|$ and Sing are left and right adjoint, respectively:*

$$|-| : \mathrm{sSets} \rightleftarrows \mathrm{Top} : \mathrm{Sing}.$$

Moreover, though we won't prove this, geometric functors canonically through the subcategory of CW complexes and cellular maps.

Definition 1.3. For any $m \geq 0$, we let $\mathrm{sk}_m X$ be the subsimplicial complex generated by the simplices of dimension at most m. Hence, $(\mathrm{sk}_m X)_n = X_n$ if $n \leq m$, and all simplices in dimensions more than m are degenerate.

Date: 18 February 2016.

Example 1.4. Let $\partial\Delta^n \subseteq \Delta^n$ be $\mathrm{sk}_{n-1}\Delta^n$ for $n \geq 1$. By convention, $\partial\Delta^0$ is decreed to be empty. Prove that $|\partial\Delta^n|$ has the weak homotopy type of S^{n-1}. We call $\partial\Delta^n$ the **boundary** of the n-simplex.

Example 1.5. Another important class of simplicial sets are the **horns**. For each $n \geq 1$ and each $0 \leq i \leq n$, we let $\Lambda^n_i \subseteq \Delta^n$ be the largest sub-simplicial set not containing $\partial_i(\iota_n)$, where $\iota_n \in \Delta^n_n$ is the non-degenerate cell classifying the identity $[n] \to [n]$. The geometric realization $|\Lambda^n_i|$ is contractible for all n and i.

2. Simplicial homology

Given a functor $F : \mathrm{Sets} \to C$ and a simplicial set X, the composition $F \circ X$ is a simplicial object in C. Of particular interest is when we consider $R[-] : \mathrm{Sets} \to \mathrm{Mod}_R$, the free R-module functor for a ring R. Applying this to X, we obtain $R[X]$ a simplicial R-module, which is moreover free in each degree. We let $\mathrm{C}(R[X])$ be the associated chain complex. Here, $\mathrm{C}_n(R[X]) = R[X_n]$, and the boundary map $d_n : \mathrm{C}_n(R[X]) \to \mathrm{C}_{n-1}(R[X])$ is given as

$$d_n = \sum_{i=0}^{n} (-1)^i \partial_i.$$

This is called the homology of X with coefficients in R, and we'll write the homology groups as $\mathrm{H}_n(X, R)$.

Lemma 2.1. *If X is a topological space, then there is a natural isomorphism* $\mathrm{C}(R[\mathrm{Sing}(X)]) \cong \mathrm{C}(X, R)$, *where* $\mathrm{C}(X, R)$ *is the usual singular chain complex computing R-homology.*

3. Model category structure

We equip the category of simplicial sets with a model category structure. Let W be the class of weak homotopy equivalences, i.e., maps $X \to Y$ of simplicial sets such that $|X| \to |Y|$ is a weak homotopy equivalence. Let C be the class of level-wise injections. Finally, let F be the class of Kan fibrations. A **Kan fibration** is a map $E \to B$ of simplicial sets satisfying the right lifting property with respect to all inclusions of horns $\Lambda^n_i \subseteq \Delta^n$. Thus, if $E \to B$ is a fibration, we can always find a dotted lift in the solid diagram

$$\begin{array}{ccc} \Lambda^n_i & \longrightarrow & E \\ \downarrow & \nearrow & \downarrow \\ \Delta^n & \longrightarrow & B. \end{array}$$

Theorem 3.1. *With these classes of morphisms,* sSets *is a model category.*

4. Quillen equivalences

Definition 4.1. Consider a pair of adjoint functors

$$F : M \rightleftarrows N : G$$

between model categories M and N. The pair is called a **Quillen pair**, or a a pair of Quillen functors, if one of the following equivalent conditions is satisfied:

- F preserves cofibrations and acyclic cofibrations;
- G preserves fibrations and acyclic fibrations.

In this case, F is also called a **left Quillen functor**, and G a **right Quillen functor**.

Quillen pairs provide a sufficient framework for a pair of adjoint functors on model categories to descend to a pair of adjoint functors on the homotopy categories.

Proposition 4.2. *Suppose that $F : M \rightleftarrows N : G$ is a pair of Quillen functors. Then, there are functors* $\mathbf{L}F : M \to \mathrm{Ho}(N)$ *and* $\mathbf{R}G : N \to \mathrm{Ho}(M)$, *each of which takes weak equivalences to isomorphisms, such that there is an induced adjunction* $\mathbf{L}F : \mathrm{Ho}(M) \rightleftarrows \mathrm{Ho}(N) : \mathbf{R}G$ *between homotopy categories.*

Proof. See [1, Theorem 9.7]. □

Remark 4.3. The familiar functors from homological algebra all arise in this way, so $\mathbf{L}F$ is called the left derived functor of F, while $\mathbf{R}G$ is the right derived functor of G. There is a recipe for computing the value of the derived functors on an arbitrary object X of M and Y of N. Specifically, $\mathbf{L}F(X)$ is weakly equivalent to $F(QX)$ where $QX \to X$ is an acyclic fibration with QX cofibrant in M. Similarly, $\mathbf{R}G(Y)$ is weakly equivalent to $G(RY)$ where $Y \to RY$ is an acyclic cofibration with RY fibrant in N.

Definition 4.4. A **Quillen equivalence** is a Quillen pair $F : M \rightleftarrows N : G$ such that $\mathbf{L}F : \mathrm{Ho}(M) \rightleftarrows \mathrm{Ho}(N) : \mathbf{R}G$ is an inverse equivalence.

Proposition 4.5. *In the situation of the previous proposition, if in addition for every morphism $f : A \to G(X)$ in M, where A is cofibrant and X is fibrant, the conditions that f and the adjoint $F(A) \to X$ are weak equivalences are equivalent, then $\mathbf{L}F$ and $\mathbf{R}G$ are inverse equivalences.*

Theorem 4.6. *Geometric realization and the singular set functor form a Quillen equivalence pair.*

References

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