548.
Lecture 24.
9 March 2015.

Def. A space X is paracompact if every open cover admits a locally finite subcover.

If X is paracompact and Hausdorff, it admits partitions of unity.

Ex. CW complexes.

Theorem. If E is a rank k vector on a paracompact Hausdorff space, there is a Gauss map $E \to \mathbb{C}^\infty$. If E is trivial on a cover of n open sets, there is a Gauss map $E \to \mathbb{C}^{nk}$.

proof. Let $\{U_i\}_{i\in I}$ be a locally finite cover on which E is trivial. Fix $\phi_i : U_i \times \mathbb{C}^k \xrightarrow{\cong} E_i$. Fix a partition of ~~unity~~ unity $\rho_i : X \to \mathbb{R}$ with $\overline{\rho_i^{-1}((0,1])} \subseteq U_i$. Let

$$\begin{array}{ccc} E & \longrightarrow & \bigoplus_{i\in I} \mathbb{C}^k \\ \pi \big\downarrow & & \big\downarrow \\ X & \longrightarrow & pt \end{array}$$

be defined by $\sum g_i$, where $g_i = (\rho_i \circ \pi)\, p_2 \circ \phi_i^{-1}$ on U_i and zero outside. Easy to see this is a Gauss map.

Cor. Every VB on a paracompact Hausdorff space is the pullback of the universal vector bundle on $Gr_k(\mathbb{C}^\infty)$.

Def. A VB $E \to X$ is of finite type if it trivializes on a finite cover.

Ex. On a finite CW complex, all VBs are of finite type.

Prop. TFAE for $E \xrightarrow{\pi} X$.

(1) E is of finite type.

(2) $E \cong f^*(\gamma_k^n)$ for some n.

(3) There is some F s.t. $E \oplus F \cong X \times \mathbb{C}^n$.

proof. Use the orthogonal complement bundle $(\gamma_k^n)^\perp$ on $Gr_k(\mathbb{C}^n)$.

Remark. Serre proved that vector bundles on a finite CW complex X all correspond to f.g. projective modules over $C(X)$.

Definition. Two vector bundles E and F on X are homotopic if there exists a v.b. $G \to X \times I$ such that $i_0^* G \cong E$ and $i_1^* G \cong F$.

Lemma. $B_1 = A \times [a,c]$, $B_2 = A \times [c,b]$. If E is a v.b. on $X = B_1 \cup B_2$ that is trivial on B_1 and B_2, then E is trivial on X.

proof. $\phi_i : B_i \times \mathbb{C}^n \cong E_i$.

$$v_i = \phi_i|_{A \times \{c\}}.$$

$$h = v_2^{-1} v_1$$

$$h(x,v) = (x, g(x)v)$$

$$g : A \to GL_n(\mathbb{C}).$$

Define $w : B_2 \times \mathbb{C}^n \cong B_2 \times \mathbb{C}^n$

$$w(x,t,v) = (x,t,g(x)v).$$

Now, ϕ_1 and $\phi_2 \circ w$ are equal on $A \times \{c\}$.

Lemma. If E is a vb on $X \times I$, then there's an open cover $\{U_i\}$ of X s.t. E is trivial on $U_i \times I$.

proof. Subdivide, use compactness of I.