Spectral sequences?

<u>Definition</u>. If $K=\{K_n\}$ is a spectrum, the [redacted] homotopy groups of $K$[redacted] are

$$\pi_i K = \operatorname*{colim}_n \pi_{i+n} K_n,$$

where $\pi_{i+n} K_n \to \pi_{i+n+1} K_{n+1}$ is induced by $K_n \to \Omega K_{n+1}$, which leads to

$$\begin{array}{ccccc} S^{i+n} & \longrightarrow & K_n & \longrightarrow & \Omega K_{n+1} \\ & & | & & \\ \Sigma S^{i+n} & & \longrightarrow & & K_{n+1} \\ \cong & & & & \\ S^{i+n+1} & & & & \end{array}$$

<u>Remark</u>. Even if $K_n = pt$ for $n<0$, then one can have $\pi_i K_n \neq 0$ for $i<0$.

<u>Exs</u>. (1) Fix $A$. Set $K_n = K(A,n)$ for $n \geq 0$, $K_n = pt$ for $n<0$. This spectrum is the <u>Eilenberg-MacLane spectrum</u> <u>$HA$</u>.

$$\pi_i HA \cong \begin{cases} A & i=0, \\ 0 & i\neq 0. \end{cases}$$

$\Omega$-spectra.

(2) Fix a pointed space $X$. Set $K_n = \Sigma^n X$ for $n\geq 0$, $pt$ $n<0$. This is the <u>suspension spectrum</u> $\Sigma^\infty X$.

$$\pi_i \Sigma^\infty X \cong \pi_i^s X.$$

(3) For $X = S^0$, one gets $\Sigma^\infty S^0 = \mathbb{S}$ (think $\mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{F}_p$), the <u>sphere</u> <u>spectrum</u>, and

$$\pi_i \mathbb{S} \cong \pi_i^s.$$

not $\Omega$-spectra.

Rem. Why are $\Omega$-spectra nice? The map

$$\pi_i K_0 \longrightarrow \pi_i K$$

is an iso. for $K = \{K_n\}_{n \in \mathbb{Z}}$ an $\Omega$-spectrum.

Rem. Why are spectra nice? We've inverted the suspension map on spaces. Suspension becomes equivalent to shifting. Define

$$(\Sigma K)_n = \Sigma K_n,$$

with the usual bonding maps. We can also construct the loop spectrum

$$(\Omega K)_n = \Omega K_n.$$

For any space, the adjunction leads to a unit map

$$X \to \Omega \Sigma X.$$

It turns out that this is a stable weak equivalence, and that at the level of spectra,

$$K \simeq \Omega \Sigma K.$$

__Theorem__. If $K$ is an $\Omega$-spectrum,

$$\tilde{h}^n(X) := [X, K_n]_*$$

is a reduced cohomology theory on pointed CW complexes.

__proof__. It's trivial to check (1) and (3). Let $(X,A)$ be a CW pair.

$$\begin{tikzcd}
A \arrow[r, hook] \arrow[d, equal] & X \arrow[r, hook] \arrow[d, equal] & X \cup CA \arrow[r, hook] \arrow[d, "\simeq"] & (X \cup CA) \cup CX \arrow[r, hook] \arrow[d, shift left, "\simeq"'] & ((X \cup CA) \cup CX) \cup C(X \cup CA) \arrow[d, shift left, "\simeq"'] \\
A \arrow[r, hook] & X \arrow[r] & X/A \arrow[r] & SA \arrow[u, shift left] \arrow[r] & SX \arrow[u, shift left] & \cdots
\end{tikzcd}$$

Cofibration sequence

$$A \hookrightarrow X \to X/A \to \Sigma A \to \Sigma X \to \Sigma(X/A) \to \cdots,$$

which is natural up to homotopy. Get exact sequence

$$\begin{tikzcd}
\cdots & {[A, K_n]_*} \arrow[l] & {[X, K_n]_*} \arrow[l] & {[X/A, K_n]_*} \arrow[l] & {[A, K_{n-1}]_*} \arrow[l] \arrow[d, phantom, "\cong" rotate=90] & {[X, K_{n-1}]_*} \arrow[l] \arrow[d, phantom, "\cong" rotate=90] & \cdots \arrow[l] \\
 & & & & {[A, \Omega K_n]_*} \arrow[d, phantom, "\cong" rotate=90] & {[X, \Omega K_n]_*} \arrow[d, phantom, "\cong" rotate=90] & \\
 & & & & {[\Sigma A, K_n]_*} & {[\Sigma X, K_n]_*} &
\end{tikzcd}$$

$\cancel{[A, \Omega K_{n+1}]} \cong \cancel{[\Sigma A, K_{n+1}]}$