Problem set 8 for 131 A/3 - Fall 2012

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1. An algebraic number in $\mathbb{R}$ is any number that is the solution to an equation of the form

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0,$$

where the a_i are rational numbers. Show that the set of algebraic numbers in $\mathbb{R}$ is countable.

2. Prove that there exist non-algebraic real numbers.

3. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function such that for each point $x \in \mathbb{R}$ there exists a $\delta > 0$ such that if $|y - x| < \delta$, then $f(y) \geq f(x)$. Show that f takes on only countably many values.

4. Is the set of all irrational numbers countable? Give a proof of your answer.

5. Show that a countable union of countable sets is countable.

6. If S is a set, its power set $P(S)$ is the set of all subsets of S. For example, $P(\emptyset) = \{\emptyset\}$. Show that S and $P(S)$ never have the same cardinality.

References

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[Ros80] K. A. Ross, *Elementary analysis: the theory of calculus*, Springer-Verlag, New York, 1980. Undergraduate Texts in Mathematics.

[Rud87] W. Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill Book Co., New York, 1987.