Midterm - Math 132/3

Due 12pm on 8 May 2012

This is the midterm for the course. You are not permitted to work on this exam with others. Your work should be your own. You may consult the course textbook and your notes. Questions may be sent to Brad or I by 5pm on Saturday to guarantee a response by the end of the weekend. We will post responses on Piazza for everyone to see, where applicable. Show all work and write as neatly as possible.

1. (20pts) Compute all values of the following complex powers: (a) $(1+i)^{1/4}$, (b) i^i, (c) $3^{1/5}$, (d) $(1+i)^{1+i}$.

2. (20pts) Verify that the following functions are harmonic and compute their harmonic conjugates on the given domains: (a) $u(x,y) = \frac{y}{x^2+y^2}$ on $\mathbb{C} - \{0\}$, (b) $u(x,y) = e^x(x\cos x - y\cos y)$ on $\mathbb{C}$.

3. (20pts) We showed in class that linear fractional transformations are 3-transitive. That is, for any 2 sets of 3 distinct points in $\mathbb{P}^1 = \mathbb{C} \cup \{\infty\}$, say $\{x_0, x_1, x_2\}$ and $\{y_0, y_1, y_2\}$, there is a linear fractional transformation $f(z) = \frac{az+b}{cz+d}$ such that $f(x_i) = y_i$ for $i = 0, 1, 2$. Are linear fractional transformations 4-transitive? That is, given 2 sets of 4 distinct points $\{x_0, x_1, x_2, x_3\}$ and $\{y_0, y_1, y_2, y_3\}$, is there a linear fractional transformation $f(z)$ such that $f(x_i) = y_i$ for $i = 0, 1, 2, 3$? If yes, prove it. If no, provide a counterexample with proof.

4. (10pts) Determine the linear fractional transformation $f = \frac{az+b}{cz+d}$ that satisfies $f(0) = 1$, $f(1) = 5$, and $f(\infty) = 3$.

5. (20pts) Compute the following line integrals: (a) $\int_\gamma x dy$, where γ is the semicircle in the upper half-plane from R to $-R$ and R is a positive real number, (b) $\int_\gamma xy^4 dx$ where γ is the right half of the circle $|z| = 4$, in the counterclockwise direction.

6. (20pts) Show that if $f = u + iv$ and $f = u - iv$ are both analytic, where u and v are real-valued functions, then f is a constant function.

7. (30pts) A polynomial $P(x, y)$ is called harmonic if it satisfies Laplace's equation. Determine all harmonic polynomials of the form $P(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$, and find their harmonic conjugates. Show that a harmonic polynomial has a harmonic conjugate on $\mathbb{C}$, and that any such harmonic conjugate is a harmonic polynomial in x and y. Then, show that for any harmonic conjugate $Q(x, y)$ of $P(x, y)$, the analytic function $f(x + iy) = P(x, y) + iQ(x, y)$ is a complex polynomial in z.

8. (30pts) Show that on the punctured unit disk $D = \{z : 0 < |z| < 1\}$ there is a non-exact closed differential $Pdx + Qdy$ such that if $Sdx + Tdy$ is a closed differential on D, then $Sdx + Tdy = \alpha(Pdx + Qdy) + dh$ for some real number α and some function h. Show that on the domain

$$E = \{z : 0 < |z| < 2 \text{ and } z \neq 0, i\}$$

there are two closed differentials $P_0dx + Q_0dy$ and $P_1dx + Q_1dy$ such that every non-exact closed differential on D is equal to $\alpha(P_0dx+Q_0dy)+\beta(P_1dx+Q_1dy) + dh$ for real numbers α and β and some function h. You make take the first differential to be $\frac{-ydx+xdy}{x^2+y^2}$. (All functions h, P_i, and Q_i are assumed to have continuous second-order partial derivatives.)