115a/4 - Practice Final

1 December 2010

1. Prove **from the axioms** of a vector space that additive inverses are unique in a vector space. In other words, prove that if $x + y = 0$ and $x + z = 0$, then $y = z$.

2. Show that $(5, 4, 3)$, $(4, 3, 2)$, and $(3, 2, 0)$ are linearly independent in $\mathbb{R}^3$.

3. Prove that $P_n(\mathbb{R})$, the $\mathbb{R}$-vector space of degree n polynomials, is finite dimensional.

4. Use the Replacement Theorem to prove that if V is a finitely generated vector space, then any two bases have the same number of elements.

5. Prove that if V is finite dimensional, and if $T : V \to W$ is a linear transformation, then

$$dim(V) = nullity(T) + rank(T),$$

the sum of the nullity and rank of T.

6. Let $\beta = \{1, x, x^2\}$, and let $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ be the linear transformation given by

$$T(f(x)) = (x - 1)f(1) + x^2 f(4).$$

Compute the matrix representation $[T]_\beta$ of T with respect to β.

7. Let $y = (1, 1, 1)$, and let $\mathbb{R}^3$ be the vector space equipped with the standard inner product. Let $g : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$g(x) =< x, y > y.$$

What are the eigenvalues of g?

8. Is the matrix

$$\begin{pmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

diagonalizable? Prove your answer.

9. Let $\beta = \{(5,4,3),(4,3,2),(3,2,0)\}$ be the ordered basis for $\mathbb{R}^3$. Compute the Gram-Schmidt orthogonalization of β with respect to the standard inner product on $\mathbb{R}^3$.

10. Let $P_3(\mathbb{R})$ be the vector space of degree 3 polynomials with coefficients in $\mathbb{R}$. Equip $P_3(\mathbb{R})$ with the inner product

$$< f(x), g(x) >= \int_{-1}^{1} f(x)g(x)dx.$$

Let W be the space generated by $x^2 + x + 1$ and x. Compute the orthogonal complement $W^{\perp}$.