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(i) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$ and $\mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$

(j) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$ and $\mathsf{T}(A) = A^t + 2 \cdot \operatorname{tr}(A) \cdot I_2$

5. Prove Theorem 5.4.

6. Let T be a linear operator on a finite-dimensional vector space V, and let β be an ordered basis for V. Prove that λ is an eigenvalue of T if and only if λ is an eigenvalue of $[\mathsf{T}]_\beta$.

7. Let T be a linear operator on a finite-dimensional vector space V. We define the **determinant** of T, denoted $\det(\mathsf{T})$, as follows: Choose any ordered basis β for V, and define $\det(\mathsf{T}) = \det([\mathsf{T}]_\beta)$.

(a) Prove that the preceding definition is independent of the choice of an ordered basis for V. That is, prove that if β and γ are two ordered bases for V, then $\det([\mathsf{T}]_\beta) = \det([\mathsf{T}]_\gamma)$.
(b) Prove that T is invertible if and only if $\det(\mathsf{T}) \neq 0$.
(c) Prove that if T is invertible, then $\det(\mathsf{T}^{-1}) = [\det(\mathsf{T})]^{-1}$.
(d) Prove that if U is also a linear operator on V, then $\det(\mathsf{TU}) = \det(\mathsf{T}) \cdot \det(\mathsf{U})$.
(e) Prove that $\det(\mathsf{T} - \lambda \mathsf{I_V}) = \det([\mathsf{T}]_\beta - \lambda I)$ for any scalar λ and any ordered basis β for V.

8. (a) Prove that a linear operator T on a finite-dimensional vector space is invertible if and only if zero is not an eigenvalue of T.
(b) Let T be an invertible linear operator. Prove that a scalar λ is an eigenvalue of T if and only if λ^{-1} is an eigenvalue of T^{-1}.
(c) State and prove results analogous to (a) and (b) for matrices.

9. Prove that the eigenvalues of an upper triangular matrix M are the diagonal entries of M.

10. Let V be a finite-dimensional vector space, and let λ be any scalar.

(a) For any ordered basis β for V, prove that $[\lambda \mathsf{I_V}]_\beta = \lambda I$.
(b) Compute the characteristic polynomial of $\lambda \mathsf{I_V}$.
(c) Show that $\lambda \mathsf{I_V}$ is diagonalizable and has only one eigenvalue.

11. A **scalar matrix** is a square matrix of the form λI for some scalar λ; that is, a scalar matrix is a diagonal matrix in which all the diagonal entries are equal.

(a) Prove that if a square matrix A is similar to a scalar matrix λI, then $A = \lambda I$.
(b) Show that a diagonalizable matrix having only one eigenvalue is a scalar matrix.

(c) Prove that $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ is not diagonalizable.

12. (a) Prove that similar matrices have the same characteristic polynomial.
(b) Show that the definition of the characteristic polynomial of a linear operator on a finite-dimensional vector space V is independent of the choice of basis for V.

13. Let T be a linear operator on a finite-dimensional vector space V over a field F, let β be an ordered basis for V, and let $A = [\mathsf{T}]_\beta$. In reference to Figure 5.1, prove the following.

(a) If $v \in \mathsf{V}$ and $\phi_\beta(v)$ is an eigenvector of A corresponding to the eigenvalue λ, then v is an eigenvector of T corresponding to λ.
(b) If λ is an eigenvalue of A (and hence of T), then a vector $y \in \mathsf{F}^n$ is an eigenvector of A corresponding to λ if and only if $\phi_\beta^{-1}(y)$ is an eigenvector of T corresponding to λ.

14.† For any square matrix A, prove that A and A^t have the same characteristic polynomial (and hence the same eigenvalues).

15.† (a) Let T be a linear operator on a vector space V, and let x be an eigenvector of T corresponding to the eigenvalue λ. For any positive integer m, prove that x is an eigenvector of T^m corresponding to the eigenvalue λ^m.
(b) State and prove the analogous result for matrices.

16. (a) Prove that similar matrices have the same trace. *Hint:* Use Exercise 13 of Section 2.3.
(b) How would you define the trace of a linear operator on a finite-dimensional vector space? Justify that your definition is well-defined.

17. Let T be the linear operator on $\mathsf{M}_{n\times n}(R)$ defined by $\mathsf{T}(A) = A^t$.

(a) Show that ± 1 are the only eigenvalues of T.
(b) Describe the eigenvectors corresponding to each eigenvalue of T.
(c) Find an ordered basis β for $\mathsf{M}_{2\times 2}(R)$ such that $[\mathsf{T}]_\beta$ is a diagonal matrix.
(d) Find an ordered basis β for $\mathsf{M}_{n\times n}(R)$ such that $[\mathsf{T}]_\beta$ is a diagonal matrix for $n > 2$.

18. Let $A, B \in \mathsf{M}_{n\times n}(C)$.

(a) Prove that if B is invertible, then there exists a scalar $c \in C$ such that $A + cB$ is not invertible. *Hint:* Examine $\det(A + cB)$.

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(g) $\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$

3. For each of the following linear operators T on a vector space V, test T for diagonalizability, and if T is diagonalizable, find a basis β for V such that $[\mathsf{T}]_\beta$ is a diagonal matrix.

(a) $\mathsf{V} = \mathsf{P}_3(R)$ and T is defined by $\mathsf{T}(f(x)) = f'(x) + f''(x)$, respectively.

(b) $\mathsf{V} = \mathsf{P}_2(R)$ and T is defined by $\mathsf{T}(ax^2 + bx + c) = cx^2 + bx + a$.

(c) $\mathsf{V} = \mathsf{R}^3$ and T is defined by

$$\mathsf{T}\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_2 \\ -a_1 \\ 2a_3 \end{pmatrix}.$$

(d) $\mathsf{V} = \mathsf{P}_2(R)$ and T is defined by $\mathsf{T}(f(x)) = f(0) + f(1)(x + x^2)$.

(e) $\mathsf{V} = \mathsf{C}^2$ and T is defined by $\mathsf{T}(z, w) = (z + iw, iz + w)$.

(f) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$ and T is defined by $\mathsf{T}(A) = A^t$.

4. Prove the matrix version of the corollary to Theorem 5.5: If $A \in \mathsf{M}_{n\times n}(F)$ has n distinct eigenvalues, then A is diagonalizable.

5. State and prove the matrix version of Theorem 5.6.

6. (a) Justify the test for diagonalizability and the method for diagonalization stated in this section.

(b) Formulate the results in (a) for matrices.

7. For

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \in \mathsf{M}_{2\times 2}(R),$$

find an expression for A^n, where n is an arbitrary positive integer.

8. Suppose that $A \in \mathsf{M}_{n\times n}(F)$ has two distinct eigenvalues, λ_1 and λ_2, and that $\dim(\mathsf{E}_{\lambda_1}) = n - 1$. Prove that A is diagonalizable.

9. Let T be a linear operator on a finite-dimensional vector space V, and suppose there exists an ordered basis β for V such that $[\mathsf{T}]_\beta$ is an upper triangular matrix.

(a) Prove that the characteristic polynomial for T splits.

(b) State and prove an analogous result for matrices.

The converse of (a) is treated in Exercise 32 of Section 5.4.

10. Let T be a linear operator on a finite-dimensional vector space V with the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ and corresponding multiplicities $m_1, m_2, \ldots, m_k$. Suppose that β is a basis for V such that $[\mathsf{T}]_\beta$ is an upper triangular matrix. Prove that the diagonal entries of $[\mathsf{T}]_\beta$ are $\lambda_1, \lambda_2, \ldots, \lambda_k$ and that each λ_i occurs m_i times $(1 \le i \le k)$.

11. Let A be an $n \times n$ matrix that is similar to an upper triangular matrix and has the distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$ with corresponding multiplicities $m_1, m_2, \ldots, m_k$. Prove the following statements.

(a) $\operatorname{tr}(A) = \displaystyle\sum_{i=1}^{k} m_i\lambda_i$

(b) $\det(A) = (\lambda_1)^{m_1}(\lambda_2)^{m_2}\cdots(\lambda_k)^{m_k}$.

12. Let T be an invertible linear operator on a finite-dimensional vector space V.

(a) Recall that for any eigenvalue λ of T, λ^{-1} is an eigenvalue of T^{-1} (Exercise 8 of Section 5.1). Prove that the eigenspace of T corresponding to λ is the same as the eigenspace of T^{-1} corresponding to λ^{-1}.

(b) Prove that if T is diagonalizable, then T^{-1} is diagonalizable.

13. Let $A \in \mathsf{M}_{n\times n}(F)$. Recall from Exercise 14 of Section 5.1 that A and A^t have the same characteristic polynomial and hence share the same eigenvalues with the same multiplicities. For any eigenvalue λ of A and A^t, let E_λ and E'_λ denote the corresponding eigenspaces for A and A^t, respectively.

(a) Show by way of example that for a given common eigenvalue, these two eigenspaces need not be the same.

(b) Prove that for any eigenvalue λ, $\dim(\mathsf{E}_\lambda) = \dim(\mathsf{E}'_\lambda)$.

(c) Prove that if A is diagonalizable, then A^t is also diagonalizable.

14. Find the general solution to each system of differential equations.

(a) $\begin{aligned} x' &= x + y \\ y' &= 3x - y \end{aligned}$ (b) $\begin{aligned} x_1' &= 8x_1 + 10x_2 \\ x_2' &= -5x_1 - 7x_2 \end{aligned}$

(c) $\begin{aligned} x_1' &= x_1 + x_3 \\ x_2' &= x_2 + x_3 \\ x_3' &= 2x_3 \end{aligned}$

15. Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Also,

$$\langle f_n, f_n\rangle = \frac{1}{2\pi}\int_0^{2\pi} e^{i(n-n)t}\,dt = \frac{1}{2\pi}\int_0^{2\pi} 1\,dt = 1.$$

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In other words, $\langle f_m, f_n\rangle = \delta_{mn}$. ♦

EXERCISES

1. Label the following statements as true or false.

(a) An inner product is a scalar-valued function on the set of ordered pairs of vectors.
(b) An inner product space must be over the field of real or complex numbers.
(c) An inner product is linear in both components.
(d) There is exactly one inner product on the vector space R^n.
(e) The triangle inequality only holds in finite-dimensional inner product spaces.
(f) Only square matrices have a conjugate-transpose.
(g) If x, y, and z are vectors in an inner product space such that $\langle x, y\rangle = \langle x, z\rangle$, then $y = z$.
(h) If $\langle x, y\rangle = 0$ for all x in an inner product space, then $y = 0$.

2. Let $x = (2, 1+i, i)$ and $y = (2-i, 2, 1+2i)$ be vectors in C^3. Compute $\langle x, y\rangle$, $\|x\|$, $\|y\|$, and $\|x+y\|$. Then verify both the Cauchy–Schwarz inequality and the triangle inequality.

3. In $\mathsf{C}([0,1])$, let $f(t) = t$ and $g(t) = e^t$. Compute $\langle f, g\rangle$ (as defined in Example 3), $\|f\|$, $\|g\|$, and $\|f+g\|$. Then verify both the Cauchy–Schwarz inequality and the triangle inequality.

4. (a) Complete the proof in Example 5 that $\langle\cdot,\cdot\rangle$ is an inner product (the Frobenius inner product) on $\mathsf{M}_{n\times n}(F)$.
(b) Use the Frobenius inner product to compute $\|A\|$, $\|B\|$, and $\langle A, B\rangle$ for

$$A = \begin{pmatrix} 1 & 2+i \\ 3 & i \end{pmatrix} \quad\text{and}\quad B = \begin{pmatrix} 1+i & 0 \\ i & -i \end{pmatrix}.$$

5. In C^2, show that $\langle x, y\rangle = xAy^*$ is an inner product, where

$$A = \begin{pmatrix} 1 & i \\ -i & 2 \end{pmatrix}.$$

Compute $\langle x, y\rangle$ for $x = (1-i, 2+3i)$ and $y = (2+i, 3-2i)$.

6. Complete the proof of Theorem 6.1.

7. Complete the proof of Theorem 6.2.

8. Provide reasons why each of the following is not an inner product on the given vector spaces.

(a) $\langle (a,b), (c,d)\rangle = ac - bd$ on R^2.
(b) $\langle A, B\rangle = \operatorname{tr}(A+B)$ on $\mathsf{M}_{2\times 2}(R)$.
(c) $\langle f(x), g(x)\rangle = \int_0^1 f'(t)g(t)\,dt$ on $\mathsf{P}(R)$, where $'$ denotes differentiation.

9. Let β be a basis for a finite-dimensional inner product space.

(a) Prove that if $\langle x, z\rangle = 0$ for all $z \in \beta$, then $x = 0$.
(b) Prove that if $\langle x, z\rangle = \langle y, z\rangle$ for all $z \in \beta$, then $x = y$.

10.† Let V be an inner product space, and suppose that x and y are orthogonal vectors in V. Prove that $\|x+y\|^2 = \|x\|^2 + \|y\|^2$. Deduce the Pythagorean theorem in R^2.

11. Prove the *parallelogram law* on an inner product space V; that is, show that

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2 \quad \text{for all } x, y \in \mathsf{V}.$$

What does this equation state about parallelograms in R^2?

12.† Let $\{v_1, v_2, \ldots, v_k\}$ be an orthogonal set in V, and let $a_1, a_2, \ldots, a_k$ be scalars. Prove that

$$\left\|\sum_{i=1}^{k} a_i v_i\right\|^2 = \sum_{i=1}^{k} |a_i|^2 \|v_i\|^2.$$

13. Suppose that $\langle\cdot,\cdot\rangle_1$ and $\langle\cdot,\cdot\rangle_2$ are two inner products on a vector space V. Prove that $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_1 + \langle\cdot,\cdot\rangle_2$ is another inner product on V.

14. Let A and B be $n\times n$ matrices, and let c be a scalar. Prove that $(A+cB)^* = A^* + \bar{c}B^*$.

15. (a) Prove that if V is an inner product space, then $|\langle x, y\rangle| = \|x\|\cdot\|y\|$ if and only if one of the vectors x or y is a multiple of the other. *Hint:* If the identity holds and $y \neq 0$, let

$$a = \frac{\langle x, y\rangle}{\|y\|^2},$$

and let $z = x - ay$. Prove that y and z are orthogonal and

$$|a| = \frac{\|x\|}{\|y\|}.$$

Then apply Exercise 10 to $\|x\|^2 = \|ay + z\|^2$ to obtain $\|z\| = 0$.

(b) Derive a similar result for the equality $\|x + y\| = \|x\| + \|y\|$, and generalize it to the case of n vectors.

16. (a) Show that the vector space H with $\langle \cdot, \cdot \rangle$ defined on page 332 is an inner product space.

(b) Let $\mathsf{V} = \mathsf{C}([0, 1])$, and define

$$\langle f, g \rangle = \int_0^{1/2} f(t)g(t)\,dt.$$

Is this an inner product on V?

17. Let T be a linear operator on an inner product space V, and suppose that $\|\mathsf{T}(x)\| = \|x\|$ for all x. Prove that T is one-to-one.

18. Let V be a vector space over F, where $F = R$ or $F = C$, and let W be an inner product space over F with inner product $\langle \cdot, \cdot \rangle$. If $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ is linear, prove that $\langle x, y \rangle' = \langle \mathsf{T}(x), \mathsf{T}(y) \rangle$ defines an inner product on V if and only if T is one-to-one.

19. Let V be an inner product space. Prove that

(a) $\|x \pm y\|^2 = \|x\|^2 \pm 2\Re\langle x, y \rangle + \|y\|^2$ for all $x, y \in \mathsf{V}$, where $\Re\langle x, y \rangle$ denotes the real part of the complex number $\langle x, y \rangle$.

(b) $|\,\|x\| - \|y\|\,| \leq \|x - y\|$ for all $x, y \in \mathsf{V}$.

20. Let V be an inner product space over F. Prove the *polar identities*: For all $x, y \in \mathsf{V}$,

(a) $\langle x, y \rangle = \frac{1}{4}\|x + y\|^2 - \frac{1}{4}\|x - y\|^2$ if $F = R$;

(b) $\langle x, y \rangle = \frac{1}{4}\sum_{k=1}^{4} i^k \|x + i^k y\|^2$ if $F = C$, where $i^2 = -1$.

21. Let A be an $n \times n$ matrix. Define

$$A_1 = \frac{1}{2}(A + A^*) \quad \text{and} \quad A_2 = \frac{1}{2i}(A - A^*).$$

(a) Prove that $A_1^* = A_1$, $A_2^* = A_2$, and $A = A_1 + iA_2$. Would it be reasonable to define A_1 and A_2 to be the real and imaginary parts, respectively, of the matrix A?

(b) Let A be an $n \times n$ matrix. Prove that the representation in (a) is unique. That is, prove that if $A = B_1 + iB_2$, where $B_1^* = B_1$ and $B_2^* = B_2$, then $B_1 = A_1$ and $B_2 = A_2$.

22. Let V be a real or complex vector space (possibly infinite-dimensional), and let β be a basis for V. For $x, y \in \mathsf{V}$ there exist $v_1, v_2, \ldots, v_n \in \beta$ such that

$$x = \sum_{i=1}^{n} a_i v_i \quad \text{and} \quad y = \sum_{i=1}^{n} b_i v_i.$$

Define

$$\langle x, y \rangle = \sum_{i=1}^{n} a_i \bar{b}_i.$$

(a) Prove that $\langle \cdot, \cdot \rangle$ is an inner product on V and that β is an orthonormal basis for V. Thus every real or complex vector space may be regarded as an inner product space.

(b) Prove that if $\mathsf{V} = \mathsf{R}^n$ or $\mathsf{V} = \mathsf{C}^n$ and β is the standard ordered basis, then the inner product defined above is the standard inner product.

23. Let $\mathsf{V} = \mathsf{F}^n$, and let $A \in \mathsf{M}_{n\times n}(F)$.

(a) Prove that $\langle x, Ay \rangle = \langle A^*x, y \rangle$ for all $x, y \in \mathsf{V}$.

(b) Suppose that for some $B \in \mathsf{M}_{n\times n}(F)$, we have $\langle x, Ay \rangle = \langle Bx, y \rangle$ for all $x, y \in \mathsf{V}$. Prove that $B = A^*$.

(c) Let α be the standard ordered basis for V. For any orthonormal basis β for V, let Q be the $n \times n$ matrix whose columns are the vectors in β. Prove that $Q^* = Q^{-1}$.

(d) Define linear operators T and U on V by $\mathsf{T}(x) = Ax$ and $\mathsf{U}(x) = A^*x$. Show that $[\mathsf{U}]_\beta = [\mathsf{T}]_\beta^*$ for any orthonormal basis β for V.

The following definition is used in Exercises 24–27.

Definition. *Let V be a vector space over F, where F is either R or C. Regardless of whether V is or is not an inner product space, we may still define a norm $\|\cdot\|$ as a real-valued function on V satisfying the following three conditions for all $x, y \in \mathsf{V}$ and $a \in F$:*

(1) $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$.

(2) $\|ax\| = |a| \cdot \|x\|$.

(3) $\|x + y\| \leq \|x\| + \|y\|$.

24. Prove that the following are norms on the given vector spaces V.

(a) $\mathsf{V} = \mathsf{M}_{m\times n}(F)$; $\quad \|A\| = \max_{i,j} |A_{ij}|$ for all $A \in \mathsf{V}$

(b) $\mathsf{V} = \mathsf{C}([0, 1])$; $\quad \|f\| = \max_{t\in[0,1]} |f(t)|$ for all $f \in \mathsf{V}$