Consider the polynomial $p(x) = 2+x-3x^2+5x^3$. We show that $\mathsf{L}_A\phi_\beta(p(x)) = \phi_\gamma \mathsf{T}(p(x))$. Now

$$\mathsf{L}_A\phi_\beta(p(x)) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -3 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -6 \\ 15 \end{pmatrix}.$$

HW 5 due 29 October.

But since $\mathsf{T}(p(x)) = p'(x) = 1 - 6x + 15x^2$, we have

$$\phi_\gamma \mathsf{T}(p(x)) = \begin{pmatrix} 1 \\ -6 \\ 15 \end{pmatrix}.$$

So $\mathsf{L}_A\phi_\beta(p(x)) = \phi_\gamma \mathsf{T}(p(x))$. ◆

Try repeating Example 7 with different polynomials $p(x)$.

EXERCISES

1. Label the following statements as true or false. In each part, V and W are vector spaces with ordered (finite) bases α and β, respectively, $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ is linear, and A and B are matrices.
   (a) $\left([\mathsf{T}]_\alpha^\beta\right)^{-1} = [\mathsf{T}^{-1}]_\alpha^\beta$.
   (b) T is invertible if and only if T is one-to-one and onto.
   (c) $\mathsf{T} = \mathsf{L}_A$, where $A = [\mathsf{T}]_\alpha^\beta$.
   (d) $\mathsf{M}_{2\times 3}(F)$ is isomorphic to F^5.
   (e) $\mathsf{P}_n(F)$ is isomorphic to $\mathsf{P}_m(F)$ if and only if $n = m$.
   (f) $AB = I$ implies that A and B are invertible.
   (g) If A is invertible, then $(A^{-1})^{-1} = A$.
   (h) A is invertible if and only if L_A is invertible.
   (i) A must be square in order to possess an inverse.

2. For each of the following linear transformations T, determine whether T is invertible and justify your answer.
   (a) $\mathsf{T}\colon \mathsf{R}^2 \to \mathsf{R}^3$ defined by $\mathsf{T}(a_1, a_2) = (a_1 - 2a_2, a_2, 3a_1 + 4a_2)$.
   (b) $\mathsf{T}\colon \mathsf{R}^2 \to \mathsf{R}^3$ defined by $\mathsf{T}(a_1, a_2) = (3a_1 - a_2, a_2, 4a_1)$.
   (c) $\mathsf{T}\colon \mathsf{R}^3 \to \mathsf{R}^3$ defined by $\mathsf{T}(a_1, a_2, a_3) = (3a_1 - 2a_3, a_2, 3a_1 + 4a_2)$.
   (d) $\mathsf{T}\colon \mathsf{P}_3(R) \to \mathsf{P}_2(R)$ defined by $\mathsf{T}(p(x)) = p'(x)$.
   (e) $\mathsf{T}\colon \mathsf{M}_{2\times 2}(R) \to \mathsf{P}_2(R)$ defined by $\mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = a + 2bx + (c+d)x^2$.
   (f) $\mathsf{T}\colon \mathsf{M}_{2\times 2}(R) \to \mathsf{M}_{2\times 2}(R)$ defined by $\mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a+b & a \\ c & c+d \end{pmatrix}$.

3. Which of the following pairs of vector spaces are isomorphic? Justify your answers.
   (a) F^3 and $\mathsf{P}_3(F)$.
   (b) F^4 and $\mathsf{P}_3(F)$.
   (c) $\mathsf{M}_{2\times 2}(R)$ and $\mathsf{P}_3(R)$.
   (d) $\mathsf{V} = \{A \in \mathsf{M}_{2\times 2}(R)\colon \operatorname{tr}(A) = 0\}$ and R^4.

4.† Let A and B be $n \times n$ invertible matrices. Prove that AB is invertible and $(AB)^{-1} = B^{-1}A^{-1}$.

5.† Let A be invertible. Prove that A^t is invertible and $(A^t)^{-1} = (A^{-1})^t$.

6. Prove that if A is invertible and $AB = O$, then $B = O$.

7. Let A be an $n \times n$ matrix.
   (a) Suppose that $A^2 = O$. Prove that A is not invertible.
   (b) Suppose that $AB = O$ for some nonzero $n \times n$ matrix B. Could A be invertible? Explain.

8. Prove Corollaries 1 and 2 of Theorem 2.18.

9. Let A and B be $n \times n$ matrices such that AB is invertible. Prove that A and B are invertible. Give an example to show that arbitrary matrices A and B need not be invertible if AB is invertible.

10.† Let A and B be $n \times n$ matrices such that $AB = I_n$.
   (a) Use Exercise 9 to conclude that A and B are invertible.
   (b) Prove $A = B^{-1}$ (and hence $B = A^{-1}$). (We are, in effect, saying that for square matrices, a "one-sided" inverse is a "two-sided" inverse.)
   (c) State and prove analogous results for linear transformations defined on finite-dimensional vector spaces.

11. Verify that the transformation in Example 5 is one-to-one.

12. Prove Theorem 2.21.

13. Let $\sim$ mean "is isomorphic to." Prove that $\sim$ is an equivalence relation on the class of vector spaces over F.

14. Let

$$\mathsf{V} = \left\{ \begin{pmatrix} a & a+b \\ 0 & c \end{pmatrix} : a, b, c \in F \right\}.$$

Construct an isomorphism from V to F^3.

15. Let V and W be n-dimensional vector spaces, and let $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ be a linear transformation. Suppose that β is a basis for V. Prove that T is an isomorphism if and only if $\mathsf{T}(\beta)$ is a basis for W.

16. Let B be an $n \times n$ invertible matrix. Define $\Phi: \mathsf{M}_{n\times n}(F) \to \mathsf{M}_{n\times n}(F)$ by $\Phi(A) = B^{-1}AB$. Prove that Φ is an isomorphism.

17.† Let V and W be finite-dimensional vector spaces and $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ be an isomorphism. Let V_0 be a subspace of V.

(a) Prove that $\mathsf{T}(\mathsf{V}_0)$ is a subspace of W.
(b) Prove that $\dim(\mathsf{V}_0) = \dim(\mathsf{T}(\mathsf{V}_0))$.

18. Repeat Example 7 with the polynomial $p(x) = 1 + x + 2x^2 + x^3$.

19. In Example 5 of Section 2.1, the mapping $\mathsf{T}: \mathsf{M}_{2\times 2}(R) \to \mathsf{M}_{2\times 2}(R)$ defined by $\mathsf{T}(M) = M^t$ for each $M \in \mathsf{M}_{2\times 2}(R)$ is a linear transformation. Let $\beta = \{E^{11}, E^{12}, E^{21}, E^{22}\}$, which is a basis for $\mathsf{M}_{2\times 2}(R)$, as noted in Example 3 of Section 1.6.

(a) Compute $[\mathsf{T}]_\beta$.
(b) Verify that $\mathsf{L}_A\phi_\beta(M) = \phi_\beta \mathsf{T}(M)$ for $A = [\mathsf{T}]_\beta$ and

$$M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

20.† Let $\mathsf{T}: \mathsf{V} \to \mathsf{W}$ be a linear transformation from an n-dimensional vector space V to an m-dimensional vector space W. Let β and γ be ordered bases for V and W, respectively. Prove that $\text{rank}(\mathsf{T}) = \text{rank}(\mathsf{L}_A)$ and that $\text{nullity}(\mathsf{T}) = \text{nullity}(\mathsf{L}_A)$, where $A = [\mathsf{T}]_\beta^\gamma$. *Hint:* Apply Exercise 17 to Figure 2.2.

21. Let V and W be finite-dimensional vector spaces with ordered bases $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\gamma = \{w_1, w_2, \ldots, w_m\}$, respectively. By Theorem 2.6 (p. 72), there exist linear transformations $\mathsf{T}_{ij}: \mathsf{V} \to \mathsf{W}$ such that

$$\mathsf{T}_{ij}(v_k) = \begin{cases} w_i & \text{if } k = j \\ 0 & \text{if } k \neq j. \end{cases}$$

First prove that $\{\mathsf{T}_{ij}: 1 \le i \le m,\ 1 \le j \le n\}$ is a basis for $\mathcal{L}(\mathsf{V}, \mathsf{W})$. Then let M^{ij} be the $m \times n$ matrix with 1 in the ith row and jth column and 0 elsewhere, and prove that $[\mathsf{T}_{ij}]_\beta^\gamma = M^{ij}$. Again by Theorem 2.6, there exists a linear transformation $\Phi: \mathcal{L}(\mathsf{V}, \mathsf{W}) \to \mathsf{M}_{m\times n}(F)$ such that $\Phi(\mathsf{T}_{ij}) = M^{ij}$. Prove that Φ is an isomorphism.

22. Let $c_0, c_1, \ldots, c_n$ be distinct scalars from an infinite field F. Define $\mathsf{T}: \mathsf{P}_n(F) \to \mathsf{F}^{n+1}$ by $\mathsf{T}(f) = (f(c_0), f(c_1), \ldots, f(c_n))$. Prove that T is an isomorphism. *Hint:* Use the Lagrange polynomials associated with $c_0, c_1, \ldots, c_n$.

23. Let V denote the vector space defined in Example 5 of Section 1.2, and let $\mathsf{W} = \mathsf{P}(F)$. Define

$$\mathsf{T}: \mathsf{V} \to \mathsf{W} \quad \text{by} \quad \mathsf{T}(\sigma) = \sum_{i=0}^{n} \sigma(i)x^i,$$

where n is the largest integer such that $\sigma(n) \neq 0$. Prove that T is an isomorphism.

The following exercise requires familiarity with the concept of *quotient space* defined in Exercise 31 of Section 1.3 and with Exercise 40 of Section 2.1.

24. Let $\mathsf{T}: \mathsf{V} \to \mathsf{Z}$ be a linear transformation of a vector space V onto a vector space Z. Define the mapping

$$\overline{\mathsf{T}}: \mathsf{V}/\mathsf{N}(\mathsf{T}) \to \mathsf{Z} \quad \text{by} \quad \overline{\mathsf{T}}(v + \mathsf{N}(\mathsf{T})) = \mathsf{T}(v)$$

for any coset $v + \mathsf{N}(\mathsf{T})$ in $\mathsf{V}/\mathsf{N}(\mathsf{T})$.

(a) Prove that $\overline{\mathsf{T}}$ is well-defined; that is, prove that if $v + \mathsf{N}(\mathsf{T}) = v' + \mathsf{N}(\mathsf{T})$, then $\mathsf{T}(v) = \mathsf{T}(v')$.
(b) Prove that $\overline{\mathsf{T}}$ is linear.
(c) Prove that $\overline{\mathsf{T}}$ is an isomorphism.
(d) Prove that the diagram shown in Figure 2.3 commutes; that is, prove that $\mathsf{T} = \overline{\mathsf{T}}\eta$.

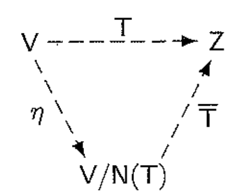

Figure 2.3

25. Let V be a nonzero vector space over a field F, and suppose that S is a basis for V. (By the corollary to Theorem 1.13 (p. 60) in Section 1.7, every vector space has a basis). Let $\mathcal{C}(S, F)$ denote the vector space of all functions $f \in \mathcal{F}(S, F)$ such that $f(s) = 0$ for all but a finite number

EXERCISES

1. Label the following statements as true or false.
   (a) Suppose that $\beta = \{x_1, x_2, \ldots, x_n\}$ and $\beta' = \{x'_1, x'_2, \ldots, x'_n\}$ are ordered bases for a vector space and Q is the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then the jth column of Q is $[x_j]_{\beta'}$.
   (b) Every change of coordinate matrix is invertible.
   (c) Let T be a linear operator on a finite-dimensional vector space V, let β and β' be ordered bases for V, and let Q be the change of coordinate matrix that changes β'-coordinates into β-coordinates. Then $[\mathsf{T}]_\beta = Q[\mathsf{T}]_{\beta'}Q^{-1}$.
   (d) The matrices $A, B \in \mathsf{M}_{n\times n}(F)$ are called similar if $B = Q^tAQ$ for some $Q \in \mathsf{M}_{n\times n}(F)$.
   (e) Let T be a linear operator on a finite-dimensional vector space V. Then for any ordered bases β and γ for V, $[\mathsf{T}]_\beta$ is similar to $[\mathsf{T}]_\gamma$.

2. For each of the following pairs of ordered bases β and β' for R^2, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.
   (a) $\beta = \{e_1, e_2\}$ and $\beta' = \{(a_1, a_2), (b_1, b_2)\}$
   (b) $\beta = \{(-1, 3), (2, -1)\}$ and $\beta' = \{(0, 10), (5, 0)\}$
   (c) $\beta = \{(2, 5), (-1, -3)\}$ and $\beta' = \{e_1, e_2\}$
   (d) $\beta = \{(-4, 3), (2, -1)\}$ and $\beta' = \{(2, 1), (-4, 1)\}$

3. For each of the following pairs of ordered bases β and β' for $\mathsf{P}_2(R)$, find the change of coordinate matrix that changes β'-coordinates into β-coordinates.
   (a) $\beta = \{x^2, x, 1\}$ and $\beta' = \{a_2x^2 + a_1x + a_0, b_2x^2 + b_1x + b_0, c_2x^2 + c_1x + c_0\}$
   (b) $\beta = \{1, x, x^2\}$ and $\beta' = \{a_2x^2 + a_1x + a_0, b_2x^2 + b_1x + b_0, c_2x^2 + c_1x + c_0\}$
   (c) $\beta = \{2x^2 - x, 3x^2 + 1, x^2\}$ and $\beta' = \{1, x, x^2\}$
   (d) $\beta = \{x^2 - x + 1, x + 1, x^2 + 1\}$ and $\beta' = \{x^2 + x + 4, 4x^2 - 3x + 2, 2x^2 + 3\}$
   (e) $\beta = \{x^2 - x, x^2 + 1, x - 1\}$ and $\beta' = \{5x^2 - 2x - 3, -2x^2 + 5x + 5, 2x^2 - x - 3\}$
   (f) $\beta = \{2x^2 - x + 1, x^2 + 3x - 2, -x^2 + 2x + 1\}$ and $\beta' = \{9x - 9, x^2 + 21x - 2, 3x^2 + 5x + 2\}$

4. Let T be the linear operator on R^2 defined by

$$\mathsf{T}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 2a + b \\ a - 3b \end{pmatrix},$$

let β be the standard ordered basis for R^2, and let

$$\beta' = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}.$$

Use Theorem 2.23 and the fact that

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

to find $[\mathsf{T}]_{\beta'}$.

5. Let T be the linear operator on $\mathsf{P}_1(R)$ defined by $\mathsf{T}(p(x)) = p'(x)$, the derivative of $p(x)$. Let $\beta = \{1, x\}$ and $\beta' = \{1 + x, 1 - x\}$. Use Theorem 2.23 and the fact that

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{pmatrix}$$

to find $[\mathsf{T}]_{\beta'}$.

6. For each matrix A and ordered basis β, find $[\mathsf{L}_A]_\beta$. Also, find an invertible matrix Q such that $[\mathsf{L}_A]_\beta = Q^{-1}AQ$.
   (a) $A = \begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$
   (b) $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$
   (c) $A = \begin{pmatrix} 1 & 1 & -1 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} \right\}$
   (d) $A = \begin{pmatrix} 13 & 1 & 4 \\ 1 & 13 & 4 \\ 4 & 4 & 10 \end{pmatrix}$ and $\beta = \left\{ \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}$

7. In R^2, let L be the line $y = mx$, where $m \neq 0$. Find an expression for $\mathsf{T}(x, y)$, where
   (a) T is the reflection of R^2 about L.
   (b) T is the projection on L along the line perpendicular to L. (See the definition of *projection* in the exercises of Section 2.1.)

8. Prove the following generalization of Theorem 2.23. Let $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ be a linear transformation from a finite-dimensional vector space V to a finite-dimensional vector space W. Let β and β' be ordered bases for

V, and let γ and γ' be ordered bases for W. Then $[\mathsf{T}]_{\beta'}^{\gamma'} = P^{-1}[\mathsf{T}]_\beta^\gamma Q$, where Q is the matrix that changes β'-coordinates into β-coordinates and P is the matrix that changes γ'-coordinates into γ-coordinates.

9. Prove that "is similar to" is an equivalence relation on $\mathsf{M}_{n\times n}(F)$.

10. Prove that if A and B are similar $n \times n$ matrices, then $\text{tr}(A) = \text{tr}(B)$. *Hint:* Use Exercise 13 of Section 2.3.

11. Let V be a finite-dimensional vector space with ordered bases α, β, and γ.
    (a) Prove that if Q and R are the change of coordinate matrices that change α-coordinates into β-coordinates and β-coordinates into γ-coordinates, respectively, then RQ is the change of coordinate matrix that changes α-coordinates into γ-coordinates.
    (b) Prove that if Q changes α-coordinates into β-coordinates, then Q^{-1} changes β-coordinates into α-coordinates.

12. Prove the corollary to Theorem 2.23.

13.† Let V be a finite-dimensional vector space over a field F, and let $\beta = \{x_1, x_2, \ldots, x_n\}$ be an ordered basis for V. Let Q be an $n \times n$ invertible matrix with entries from F. Define

$$x'_j = \sum_{i=1}^{n} Q_{ij} x_i \quad \text{for } 1 \le j \le n,$$

and set $\beta' = \{x'_1, x'_2, \ldots, x'_n\}$. Prove that β' is a basis for V and hence that Q is the change of coordinate matrix changing β'-coordinates into β-coordinates.

14. Prove the converse of Exercise 8: If A and B are each $m \times n$ matrices with entries from a field F, and if there exist invertible $m \times m$ and $n \times n$ matrices P and Q, respectively, such that $B = P^{-1}AQ$, then there exist an n-dimensional vector space V and an m-dimensional vector space W (both over F), ordered bases β and β' for V and γ and γ' for W, and a linear transformation $\mathsf{T}\colon \mathsf{V} \to \mathsf{W}$ such that

$$A = [\mathsf{T}]_\beta^\gamma \quad \text{and} \quad B = [\mathsf{T}]_{\beta'}^{\gamma'}.$$

Hints: Let $\mathsf{V} = \mathsf{F}^n$, $\mathsf{W} = \mathsf{F}^m$, $\mathsf{T} = \mathsf{L}_A$, and β and γ be the standard ordered bases for F^n and F^m, respectively. Now apply the results of Exercise 13 to obtain ordered bases β' and γ' from β and γ via Q and P, respectively.

2.6* DUAL SPACES

In this section, we are concerned exclusively with linear transformations from a vector space V into its field of scalars F, which is itself a vector space of dimension 1 over F. Such a linear transformation is called a **linear functional** on V. We generally use the letters $\mathsf{f}, \mathsf{g}, \mathsf{h}, \ldots$ to denote linear functionals. As we see in Example 1, the definite integral provides us with one of the most important examples of a linear functional in mathematics.

Example 1

Let V be the vector space of continuous real-valued functions on the interval $[0, 2\pi]$. Fix a function $g \in \mathsf{V}$. The function $\mathsf{h}\colon \mathsf{V} \to R$ defined by

$$\mathsf{h}(x) = \frac{1}{2\pi}\int_0^{2\pi} x(t)g(t)\,dt$$

is a linear functional on V. In the cases that $g(t)$ equals $\sin nt$ or $\cos nt$, $h(x)$ is often called the **nth Fourier coefficient of x.** ◆

Example 2

Let $\mathsf{V} = \mathsf{M}_{n\times n}(F)$, and define $\mathsf{f}\colon \mathsf{V} \to F$ by $\mathsf{f}(A) = \text{tr}(A)$, the trace of A. By Exercise 6 of Section 1.3, we have that f is a linear functional. ◆

Example 3

Let V be a finite-dimensional vector space, and let $\beta = \{x_1, x_2, \ldots, x_n\}$ be an ordered basis for V. For each $i = 1, 2, \ldots, n$, define $\mathsf{f}_i(x) = a_i$, where

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}$$

is the coordinate vector of x relative to β. Then f_i is a linear functional on V called the **ith coordinate function with respect to the basis β.** Note that $\mathsf{f}_i(x_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta. These linear functionals play an important role in the theory of dual spaces (see Theorem 2.24). ◆

Definition. *For a vector space* V *over* F, *we define the* **dual space** *of* V *to be the vector space* $\mathcal{L}(\mathsf{V}, F)$, *denoted by* V^*.

Thus V^* is the vector space consisting of all linear functionals on V with the operations of addition and scalar multiplication as defined in Section 2.2. Note that if V is finite-dimensional, then by the corollary to Theorem 2.20 (p. 104)

$$\dim(\mathsf{V}^*) = \dim(\mathcal{L}(\mathsf{V}, F)) = \dim(\mathsf{V}) \cdot \dim(F) = \dim(\mathsf{V}).$$