115a/4 - Homework 2*

Due 1 October 2010

1. (1.2.2) Write the zero vector of $M_{3\times 4}(F)$.

2. (1.2.11) Let $V = \{0\}$ consist of a single vector 0 and define $0 + 0 = 0$ and $c0 = 0$ for all scalars c in F. Prove that V is an F-vector space.

3. (1.2.12) A real-valued function f defined on the real line is called an even function if $f(-t) = f(t)$ for each real number t. Prove that the set of even functions defined on the real line with the operations of addition, given by $(f + g)(t) = f(t) + g(t)$, and multiplication, given by $(cf)(t) = cf(t)$, is a vector space.

4. Prove that if V is a vector space over the real numbers, then it is a vector space over the rational numbers.

5. (1.2.18) Let $V = \{(a, b) : a, b \in F\}$. Define $(a, b) + (c, d) = (a + 2c, b + 3d)$, and $c(a, b) = (ca, cb)$. Is V an F-vector space? If not, why not?

6. (1.3.2) Compute the transposes of the following matrices. If the matrix is square, compute its trace.

   (a) $\begin{pmatrix} -4 & 2 \\ 5 & -1 \end{pmatrix}$

   (b) $\begin{pmatrix} 0 & 3 & -6 \\ 2 & 4 & 7 \end{pmatrix}$

   (c) $\begin{pmatrix} -3 & 9 \\ 0 & -2 \\ 6 & 1 \end{pmatrix}$

   (d) $\begin{pmatrix} 10 & 0 & -8 \\ 2 & -4 & 3 \\ -5 & 7 & 6 \end{pmatrix}$

   (e) $\begin{pmatrix} 1 & -1 & 3 & 5 \end{pmatrix}$

   (f) $\begin{pmatrix} -2 & 5 & 1 & 4 \\ 7 & 0 & 1 & -6 \end{pmatrix}$

   (g) $\begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix}$

   (h) $\begin{pmatrix} -4 & 0 & 6 \\ 0 & 1 & -3 \\ 6 & -3 & 5 \end{pmatrix}$

7. (1.3.3) Prove that $(aA + bB)^t = aA^t + bB^t$ for any $A, B \in M_{m\times n}(F)$ and any $a, b \in F$.

*Numbers in parentheses like (1.2.11) refer to the 11th problem in the second section of the first chapter of Friedberg *et. al.*

8. (1.3.4) Prove that $(A^t)^t = A$ for each $A \in M_{m\times n}(F)$.

9. (1.3.12) An $m \times n$ matrix A is called upper triangular if $A_{ij} = 0$ whenever $i > j$. Prove that the upper triangular matrices form a subspace of $M_{m\times n}(F)$.

10. (1.3.17) Prove that a subset W of a vector space V is a subspace of V if and only if W is not empty, and, whenever $a \in F$ and $x, y \in W$, then $ax \in W$ and $x + y \in W$.

11. Prove that if U is a subspace of W and W is a subspace of V, then U is a subspace of V.

12. (1.4.3) For each of the following lists of vectors in $\mathbb{R}^3$, determine whether the first vector can be expressed as a linear combination of the other two. If it can be, find one such expression.

    (a) $(-2, 0, 3)$, $(1, 3, 0)$, $(2, 4, -1)$.
    (b) $(1, 2, -3)$, $(-3, 2, 1)$, $(2, -1, -1)$.
    (c) $(3, 4, 1)$, $(1, -2, 1)$, $(-2, -1, 1)$.
    (d) $(2, -1, 0)$, $(1, 2, -3)$, $(1, -3, 2)$.
    (e) $(5, 1, -5)$, $(1, -2, -3)$, $(-2, 3, -4)$.
    (f) $(-2, 2, 2)$, $(1, 2, -1)$, $(-3, -3, 3)$.

13. (1.4.4) For each list of polynomials in $P_3(\mathbb{R})$, determine whether the first polynomial can be expressed as a linear combination of the other two.

    (a) $x^3 - 3x + 5$, $x^3 + 2x^2 - x + 1$, $x^3 + 3x^2 - 1$.
    (b) $4x^3 + 2x^2 - 6$, $x^3 - 2x^2 + 4x + 1$, $3x^3 - 6x^2 + x + 4$.
    (c) $-2x^3 - 11x^2 + 3x + 2$, $x^3 - 2x^2 + 3x - 1$, $2x^3 + x^2 + 3x - 2$.

14. (1.4.6) Show that the vectors $(1, 1, 0)$, $(1, 0, 1)$, and $(0, 1, 1)$ generate F^3.

15. (1.4.12) Show that a subset W of a vector space V is a subspace of V if and only if $span(W) = W$.

16. (1.4.17) Let W be a subspace of an $\mathbb{R}$-vector space V. Under what conditions are there only a finite number of distinct subsets S of W such that S generates W.