Some open problems in the K-theory of ring spectra

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1 Introduction

In June of 2015 I organized a session at the Hausdorf Research Institute for Mathematics as part of the trimester program on homotopy theory, manifolds, and field theories aimed at gathering open problems about the K-theory of ring spectra. These are (rough) notes from that event, produced as faithfully as I could manage. In fact, I don't have a record of which problems from my own list I actually posed to the group, so what follows includes problems I know of, even if they were not presented, as well as the problems contributed by others.

The problems below are organized roughly into five sections: the chromatic picture, redshift, descent, geometry, and other.

I'd like to thank everyone who attended for their participation and HIM for its hospitality during my visit.

This file will be kept up-to-date when possible.

Finally, I should say that I claim no originality in the material presented below.

Conventions Connective K-theory of $\mathscr{C}$ will be $\mathrm{K}(\mathscr{C})$, while $\mathbb{K}(\mathscr{C})$ will denote nonconnective K-theory.

2 The chromatic picture

The main conjecture relating algebraic K-theory to the chromatic conjecture is the so-called Waldhausen chromatic convergence conjecture, which is to give a K-theoretic analogue of the equivalence

$$\mathbb{S}_{(p)} \simeq \operatorname*{holim}_n \mathrm{L}_n \mathbb{S}$$

proved by Hopkins and Ravenel [3, Section 8.6].

Conjecture 2.1 (Chromatic convergence conjecture)**.** *The natural map*

$$\mathrm{K}(\mathbb{S}_{(p)}) \to \operatorname*{holim}_n \mathrm{K}(\mathrm{L}_n \mathbb{S})$$

is an equivalence, at least upon taking the connective cover of the right-hand side.

This statement was not actually conjectured by Waldhausen in [5]. In fact, it was not even this tower that was studied by Waldhausen. Instead, he considered the tower involving finite localizations.

Conjecture 2.2 (Alternate chromatic convergence conjecture)**.** *The natural map*

$$\mathrm{K}(\mathbb{S}_{(p)}) \to \operatorname*{holim}_n \mathrm{K}(\mathrm{L}^f_n \mathbb{S})$$

is an equivalence, at least upon taking the connective cover of the right-hand side.

McClure and Staffeldt proved in [2] a version of Conjecture 2.1: the natural map

$$\mathrm{K}(\mathbb{S}_{(p)}) \to \operatorname*{holim}_{n} \mathrm{K}(\tau_{\geq 0}\mathrm{L}_n\mathbb{S})$$

is an equivalence.

There are fiber sequences appropriate to both K-theory towers.

Theorem 2.3. *There are fiber sequences*

$$\mathrm{K}(\mathrm{M}_n) \to \mathrm{K}(\mathrm{L}_n\mathbb{S}) \to \mathrm{K}(\mathrm{L}_{n-1}\mathbb{S})$$

for $n \geq 1$, where M_n denotes the nth monochromatic layer (at the prime p). Specifically, M_n can be identified with the compact objects in $\mathrm{L}_{\mathrm{K}(n)}\mathrm{Mod}_{\mathbb{S}}$.

Theorem 2.4. *There are fiber sequences*

$$\mathrm{K}(E) \to \mathrm{K}(\mathrm{L}_n^f\mathbb{S}) \to \mathrm{K}(\mathrm{L}_{n-1}^f\mathbb{S})$$

for $n \geq 1$, where E is the endomorphism algebra spectrum

$$\mathrm{End}_{\mathrm{L}_n^f\mathbb{S}}(\mathrm{L}_n^f\mathrm{F}(n)).$$

Here, $\mathrm{F}(n)$ denotes a finite type n spectrum.

Basically nothing is known about the Waldhausen's conjecture. I think it would be illuminated to study various parts of the problem.

Problem 2.5. *Prove that Waldhausen's conjecture is true in* rational *K-theory.*

For example, since M_n is a p-torsion category, one might wonder if $\mathrm{K}(\mathrm{M}_n)_{\mathbb{Q}} \simeq 0$ for $n \geq 1$.

Problem 2.6. *Study the first interesting fiber sequence:*

$$\mathrm{K}(\mathrm{M}_1) \to \mathrm{K}(\mathrm{L}_1\mathbb{S}) \to \mathrm{K}(\mathbb{Q}).$$

The first two terms are the K-theories of nonconnective *rings. Nevertheless, perhaps something interesting can be found by using trace methods.*

3 Redshift

While redshift is also about chromatic phenomena, it predicts a much closer connection between the chromatic level of a ring spectrum A and of $\mathrm{K}(A)$. It is related to Waldhausen's conjecture, but I don't think there are strong implications in either direction. I don't know exactly what form the so-called redshift conjecture should take. An account of Rognes can be found in [4]. One idea is the following.

Conjecture 3.1. *If $\mathrm{L}_{\mathrm{K}(n)}A \neq 0$, then $\mathrm{L}_{\mathrm{K}(n+1)}\mathrm{K}(A) \neq 0$.*

Question 3.2 (Contributed by Akhil Mathew)**.** What is the chromatic height of $\mathrm{K}(\tau_{\leq n}\mathbb{S})$?

Question 3.3. Is $\mathrm{K}^{(n)}(\mathbb{Z})$ of height n? Here, $\mathrm{K}^{(n)}$ denotes n-times itereated K-theory.

Ausoni has proved [1] that $\mathrm{L}_{\mathrm{K}(3)}\mathrm{K}(\mathrm{KU}) \simeq 0$ for $p \geq 5$. This was proved by showing that $\mathrm{K}(\mathrm{KU})$ supports a v_2-self map. Mathew-Naumann-Noel showed (in forthcoming work) that $\mathrm{L}_{\mathrm{K}(3)}\mathrm{K}(\mathrm{KU}) \simeq 0$ for $p = 2, 3, 5$ by showing that KU satisfies Artin induction for rank 2 elementary abelian p-groups at these primes.

Question 3.4. Does the fact that KU satisfies Artin induction for rank 2 elementary abelian p-groups but not rank 1 elementary abelian p-groups imply that $\mathrm{L}_{\mathrm{K}(2)}\mathrm{K}(\mathrm{KU})$ is not contractible?

4 Descent

There have been several recent advances in Galois descent for $\mathrm{K}(n)$-local algebraic K-theory of ring spectra. These will appear in upcoming papers of Clausen and of Mathew-Naumann-Noel. A major outstanding problem is to prove *hyperdescent*. We pose just one question, asking for a hyperdescent version of a theorem of Mathew-Naumann-Noel.

Question 4.1. Does $\mathrm{L}_{\mathrm{K}(n)}\mathrm{K}$ satisfy Galois hyperdescent?

5 Geometry

Answering the following question is known, by work of Farrel-Jones, to be related to pseudo-isotopy of negatively curved Riemannian manifolds. It seems especially appropriate given the recent computational advances of Blumberg and Mandell on $\mathrm{K}(\mathbb{S})$.

Question 5.1 (Contributed by Andrew Blumberg)**.** What is $\mathrm{A}(S^1) \simeq \mathrm{K}(\Sigma^\infty_+ \Omega S^1) \simeq \mathrm{K}(\mathbb{S}[t^{\pm 1}])$?

Conjecture 5.2 (Farrel-Jones Conjecture)**.** *Let G be a discrete group. The natural map*

$$\mathrm{K}(\mathbb{S}[G]) \to \operatorname*{holim}_{G/H} \mathrm{K}(\mathbb{S}[H])$$

is an equivalence, where the colimit is over the orbit category of the virtually cyclic subgroups H of G.

6 Other

It is known that a positive answer to the following question would have somewhat mysterious number-theoretic consequences.

Question 6.1 (Contributed by Andrew Blumberg)**.** Is there an integral version of TC? It should specifically satisfy the following properties:

1. $(\mathrm{TC}^{\mathrm{new}})^{\wedge}_p \simeq \mathrm{TC}^{\mathrm{old}}_p$;

2. there should be a trace map $\mathrm{K} \to \mathrm{TC}^{\mathrm{new}}$;

3. $\mathrm{TC}^{\mathrm{new}}(\mathbb{S})$ should be $\mathbb{S} \vee \mathbb{CP}^\infty_{-1}$, where $\mathbb{CP}^\infty_{-1}$ denotes the Thom spectrum of the tautological bundle on $\mathbb{CP}^\infty$.

Finally, someone asked (I'm afraid I forget who) the following question.

Question 6.2. Is it possible to use the equivariant homotopy theory to make it easier to compute $\mathrm{K}(\mathrm{Sp}_G)$?

References

[1] C. Ausoni, *On the algebraic K-theory of the complex K-theory spectrum*, Invent. Math. **180** (2010), no. 3, 611–668.

[2] J. E. McClure and R. E. Staffeldt, *The chromatic convergence theorem and a tower in algebraic K-theory*, Proc. Amer. Math. Soc. **118** (1993), no. 3, 1005–1012.

[3] D. C. Ravenel, *Nilpotence and periodicity in stable homotopy theory*, Annals of Mathematics Studies, vol. 128, Princeton University Press, Princeton, NJ, 1992. Appendix C by Jeff Smith.

[4] J. Rognes, *Chromatic redshift*, ArXiv e-prints (2014), available at http://arxiv.org/abs/1403.4838.

[5] F. Waldhausen, *Algebraic K-theory of spaces, localization, and the chromatic filtration of stable homotopy*, Algebraic topology, Aarhus 1982 (Aarhus, 1982), Lecture Notes in Math., vol. 1051, Springer, Berlin, 1984, pp. 173–195.