Rational Homotopy Theory - Lecture 18

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1. Commutative dgas as a (co)simplicial model category

A simplicial set X is **finite** if it has only finitely many non-degenerate simplices. It is **of finite type** X_n is finite for all $n \geq 0$. Similarly, we say that a cdga $R \in \mathrm{cdga}_{\mathbb{Q}}^{\geq 0}$ is **of finite type** if $\mathrm{H}^n(R)$ is finite-dimensional (over $\mathbb{Q}$) for all $n \geq 0$.

Exercise 1.1. This one is slightly harder. It is [2, Lemma 5.2]. Show that there is a natural map

$$\mathrm{Hom}_{\mathrm{cdga}}(R, \mathrm{A}^*(X) \otimes S) \to \mathrm{Hom}_{\mathrm{sSets}}(X, \mathrm{map}_{\mathrm{cdga}}(R, S)),$$

which is moreover an equivalence if either X is finite or S is of finite type.

Proposition 1.2. *The model category* cdga *satisfies axiom* **SM7**. *That is, if $i : V \to W$ is a cofibration and $p : X \to Y$ is a fibration in* cdga*, then the natural map* $\mathrm{map}(W, X) \to \mathrm{map}(W, Y) \times_{\mathrm{map}(V,Y)} \mathrm{map}(V, X)$ *is a fibration, which is acyclic if either i or p is.*

Proof. This is a proof by smell. Follow your nose from the definitions to the conclusion. □

Corollary 1.3.
(1) *If $R \to S$ is a cofibration in* cdga *and T is arbitrary, then* $\mathrm{map}_{\mathrm{cdga}}(S, T) \to \mathrm{map}_{\mathrm{cdga}}(R, T)$ *is a Kan fibration, which is a weak equivalence if i is.*
(2) *If R is cofibrant, then* $\mathrm{map}_{\mathrm{cdga}}(R, T)$ *is a Kan complex for any T.*
(3) *If R is cofibrant and $p : T \to U$ is a fibration, then* $\mathrm{map}_{\mathrm{cdga}}(R, T) \to \mathrm{map}_{\mathrm{cdga}}(R, U)$ *is a Kan fibration which is a weak equivalence if p is.*
(4) *If W is cofibrant and $f : X \to Y$ is a weak equivalence, then* $f_* : \mathrm{map}(W, X) \to \mathrm{map}(W, Y)$ *is a weak equivalence.*

Proof. Left as an exercise. □

We can also talk about mapping spaces for augmented cdgas. Given augmentated cdgas $X \to \mathbb{Q}$ and $Y \to \mathbb{Q}$, we define $\mathrm{map}_*(X, Y)$ as the fiber of $\mathrm{map}(X, Y) \to \mathrm{map}(X, k)$ over the augmentation of X.

Exercise 1.4. Show that $\mathrm{map}_*(X, Y)_p = \mathrm{Hom}_{\mathrm{cdga}_{/\mathbb{Q}}}(X, \nabla(p, *)\tilde{\otimes} Y)$, where $\nabla(p, *)\tilde{\otimes} Y \subseteq \nabla(p, *) \otimes Y$ is is the subalgebra $\mathbb{Q} \otimes \mathbb{Q} \oplus \nabla(p, *) \otimes \overline{X}$.

2. Homotopies of maps of cdgas

We define homotopies of maps of cdgas as right homotopies, and we note that

$$X \xrightarrow{\simeq} \nabla(1, *) \otimes X \to X \times X$$

is a path object for X. The first map is $x \mapsto 1 \otimes x$, while the second sends $f(t) \otimes x$ to $(f(0)x, f(1)x)$ and $g(t)dt \otimes x$ to 0. The second map is clearly surjective by looking at $t \otimes x$ and $(t - 1) \otimes x$. So, it is a fibration. It follows that $\nabla(1, *) \otimes X$ is a good path object for X, and it is actually very good because $X \to \nabla(1, *) \otimes X$ is a cofibration because $\nabla(1, *)$ is cofibrant.

It follows that the huge array of results on right homotopies work very well by using $\nabla(1, *)$. Hence, we say that two maps $f, g : X \to Y$ of cdgas are homotopic if there is a map $h : X \to \nabla(1, *) \otimes Y$ such that $\partial_0 \circ h = f$ and $\partial_1 \circ h = g$.

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For example, if X is cofibrant, then right homotopy is an equivalence relation on $\mathrm{Hom}_{\mathrm{cdga}}(X,Y)$. Everything works just as nicely for augmented cdgas. Note that by definition of the model category structure on $\mathrm{cdga}^{\geq 0}_{\mathbb{Q}/}$, an augmented cdga is cofibrant if and only if the underlying cdga is cofibrant.

Recall that if X is an augmented cdga, then $\overline{X}$ denotes the kernel of $X \to \mathbb{Q}$. We let $\mathrm{Q}X = \overline{X}/\overline{X}\cdot\overline{X}$, the chain complex of indecomposables in X. The **homotopy groups** of X are

$$\pi^n X = \mathrm{H}^n(\mathrm{Q}X).$$

We put off for some time studying how these behave with respect to homotopies in X. After our long model category theoretic detour, we can finally return to this problem.

3. Two special augmented cdgas

We let $U(n)$ be the square-zero extension of $\mathbb{Q}$ by the complex $\mathbb{Q} \xrightarrow{\mathrm{id}} \mathbb{Q}$ in degrees $n-1$ and n. Hence, $\overline{U}(n)\cdot\overline{U}(n) = 0$. Note that $U(n)$ is a quotient of $D(n)$. Similarly, let $V(n)$ be the square-zero extension of $\mathbb{Q}$ by $\mathbb{Q}$ in degree n. Again, $V(n)$ is a quotient of $S(n)$. We view these mainly as augmented cdgas.

Proposition 3.1. *If $f, g : X \to Y$ are right homotopic in augmented cdgas, then $f_* = g_* : \pi^* X \to \pi^* Y$. If $f : X \to Y$ is a weak equivalence, and if X and Y are cofibrant, then $f_* : \pi^* X \to \pi^* Y$ is an isomorphism.*

Proof. The second statement follows from the first, bearing in mind that weak equivalences between cofibrant augmented cdgas are homotopy equivalence. The first statement follows by looking at

$$\partial_0, \partial_1 : \mathrm{H}^*(\nabla(1,*)\otimes \mathrm{Q}Y) \to \mathrm{H}^*(\nabla(0,*)\otimes \mathrm{Q}Y)$$

and

$$\partial_0, \partial_1 : \mathrm{H} * \mathrm{Q}(\nabla(1,*)\tilde{\otimes} Y) \to \mathrm{H}^*\mathrm{Q}(\nabla(0,*)\tilde{\otimes} Y).$$

□

Lemma 3.2. *If X is an augmented cdga, then*

1. $\mathrm{Hom}_*(X, U(n)) \cong \mathrm{Hom}_{\mathbb{Q}}(\mathrm{Q}X^n, \mathbb{Q})$,
2. *right homotopy is an equivalence relation on* $\mathrm{Hom}_*(X, V(n))$*, and*
3. $[X, V(n)]_* \cong \mathrm{Hom}_{\mathbb{Q}}(\pi^n X, \mathbb{Q})$.

Proof. Note that there is a natural map $\mathrm{Hom}_*(X, U(n)) \to \mathrm{Hom}_{\mathbb{Q}}(\mathrm{Q}X^n, \mathbb{Q})$, given by taking $f : X \to U(n)$ to $\mathrm{Q}^n f : \mathrm{Q}^n X \to \mathrm{Q}^n U(n) \cong \mathbb{Q}$. It is easy to see that this map is surjective, and injectivity is even easier. This proves (1).

We'll see the rest of the proof next time. □

References

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