Rational Homotopy Theory - Lecture 7

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1. Model categories

Much of this section is borrowed from a forthcoming expository article I am writing with Elden Elmanto on $\mathbb{A}^1$-homotopy theory.

Model categories are a technical framework for working up to homotopy. The axioms guarantee that certain category-theoretic localizations exist without enlarging the universe and that it is possible in some sense to compute the hom-sets in the localization. The theory generalizes the use of projective or injective resolutions in the construction of derived categories of rings or schemes.

References for this material include Quillen's original book on the theory [4], Dwyer-Spalinski [1], Goerss-Jardine [2], and Goerss-Schemmerhorn [3]. For consistency, we refer the reader where possible to [2]. However, unlike some of these references, we assume that the category underlying M has all small limits and colimits. This is satisfied immediately in all cases of interest to us.

Definition 1.1. Let M be a category with all small limits and colimits. A model category structure on M consists of three classes W, C, F of morphisms in M, called **weak equivalences**, **cofibrations**, and **fibrations**, subject to the following set of axioms.

- **M1** Given $X \xrightarrow{f} Y \xrightarrow{g} Z$ two composable morphisms in M, if any two of $g \circ f$, f, and g are weak equivalences, then so is the third.
- **M2** Each class W, C, F is closed under retracts.
- **M3** Given a diagram

$$\begin{array}{ccc} Z & \longrightarrow & E \\ {\scriptstyle i}\downarrow & \nearrow & \downarrow{\scriptstyle p} \\ X & \longrightarrow & B \end{array}$$

  of solid arrows, a dotted arrow can be found making the diagram commutative if either
  - (a) p is an **acyclic fibration** ($p \in W \cap F$) and i is a cofibration, or
  - (b) i is an **acyclic cofibration** ($i \in W \cap C$) and p is a fibration.

  (In particular, cofibrations i have the **left lifting property** with respect to acyclic fibrations, while fibrations p have the **right lifting property** with respect to acyclic cofibrations.)
- **M4** Any map $X \to Z$ in M admits two factorizations $X \xrightarrow{f} E \xrightarrow{p} Z$ and $X \xrightarrow{i} Y \xrightarrow{g} Z$, such that f is an acyclic cofibration, p is a fibration, i is a cofibration, and g is an acyclic fibration.

Remark 1.2. In practice, a model category is determined by only W and either C or F. Indeed, C is precisely the class of maps in M having the left lifting property with respect to acyclic fibrations. Similarly, F consists of exactly those maps in M having the right lifting property with respect to acyclic cofibrations. The reader can prove this fact using the axioms or refer to [1, Proposition 3.13]. However, some caution is required. While one often sees model categories specified in the literature by just fixing W and either C or F, it usually has to be checked that these really do give M a model category structure.

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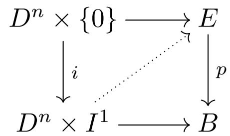
A map $p : E \to B$ is a **Serre fibration** if for every diagram



of solid arrows, a dotted arrow exists making the diagram commute. We say that $i : Z \to X$ is a **generalized relative CW complex** if Z is obtained from Z by attaching cells in no particular order of dimension.

Example 1.3. Let Top be the category of topological spaces, let W be the class of weak homotopy equivalences, C the class of maps given by retracts of generalized relative CW complexes, and F the class of Serre fibrations. Then, W, C, F is a model category structure on Top.

Remark 1.4. The acyclic Serre fibrations are those that have the RLP with respect to the inclusions $S^{n-1} \to D^n$.

Example 1.5. A slightly more natural model category structure on $\mathrm{Ch}_A^{\geq 0}$ has the same weak equivalences, but the cofibrations are the degree-wise monomorphisms in *positive degrees*. The fibrations are the surjective maps with degree-wise injective kernels. This is the **injective model category structure**.

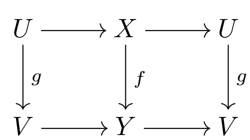
Example 1.6. Dually, on the category $\mathrm{Ch}_{\geq 0}$ of non-negatively graded *chain* complexes, there is the **projective model category structure**, with quasi-isomorphisms as weak equivalences and degree-wise surjections in positive degrees as fibrations. The cofibrations are degree-wise monomorphisms with split projective cokernels.

Example 1.7. Let A be an associative ring. Consider $\mathrm{Ch}_A^{\geq 0}$, the category of non-negatively graded cochain complexes of right A-modules. Since limits and colimits of chain complexes are computed degree-wise, $\mathrm{Ch}_A^{\geq 0}$ is closed under all small limits and colimits. Let W be the class of quasi-isomorphisms, i.e., those maps $f : M \to N$ of cochain complexes such that $\mathrm{H}^n(f) : \mathrm{H}^n(M) \to \mathrm{H}^n(N)$ is an isomorphism for all $n \geq 0$. Let F be the class of degree-wise surjections. The cofibrations are the maps having the LLP with respect to all acyclic fibrations.

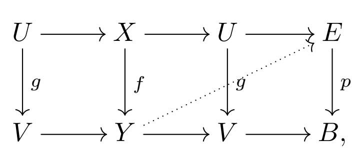
Proposition 1.8. *The class of morphisms W, C, F described in the previous example give a model category structure on* $\mathrm{Ch}_A^{\geq 0}$.

Proof. As mentioned above, limits and colimits in cochain complexes are no problem. Axiom **M1** is trivial to verify, since isomorphisms of abelian groups satisfy the two-out-of-three property.

A retract of a morphism $f : X \to Y$ is another morphism $g : U \to V$ fitting into a commutative diagram



such that the compositions $U \to U$ and $V \to V$ are the identities id_U and id_V, respectively. An easy diagram chase shows that if f is a quasi-isomorphism, then so is g. Similarly, the fibrations are closed under retracts by an easy diagram chase. Now, consider a retract $g : U \to V$ of a cofibration $X \to Y$, and consider the following test diagram of solid arrows:



where p is an acyclic fibration, and where the dotted arrow exists since f is a cofibration Since the compositions $U \to U$ and $V \to V$ are the identity, the composition $V \to Y \to E$ is the desired lift. Hence, g is a cofibration.

We will verify **M4** before **M3**, as we use factorizations in the proof. So, suppose that $h : X \to Z$ is an arbitrary map of chain complexes. It's useful to consider some **elementary cofibrations**. So, we let $D(n)$ be the cochain complex $A \xrightarrow{\mathrm{id}_A} A$ in degrees $n-1$ and n, and zero elsewhere. Clearly, $\mathrm{H}^*(D(n)) = 0$. Similarly, we let $S(n)$ the cochain complex A concentrated in degree n, and 0 elsewhere. We have $\mathrm{H}^m(S(n)) = A$ if $m = 0$ and 0 otherwise.

Note that isomorphisms are cofibrations, that compositions of infinite sequences of cofibrations are cofibrations, that pushouts of cofibrations are cofibrations, and that coproducts of cofibrations are cofibrations.

Set

$$Y_f = X \oplus_{z \in Z} D(|z| + 1).$$

The map $X \to Y_f \to Z$ is an acyclic cofibration followed by a fibration. This proves one part of **M4**. The rest of the proof is left until next time. □

References

[1] W. G. Dwyer and J. Spaliński, *Homotopy theories and model categories*, Handbook of algebraic topology, North-Holland, Amsterdam, 1995, pp. 73–126.

[2] P. G. Goerss and J. F. Jardine, *Simplicial homotopy theory*, Progress in Mathematics, vol. 174, Birkhäuser Verlag, Basel, 1999.

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[4] D. G. Quillen, *Homotopical algebra*, Lecture Notes in Mathematics, No. 43, Springer-Verlag, Berlin-New York, 1967.