Rational Homotopy Theory - Lecture 5

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1. More about minimal cdgas

Let A be strictly connected. Recall that $\mathrm{Q}A = A^+/(A^+ \cdot A^+)$ is the cochain complex of indecomposables in A, where A^+ is the kernel of the augmentation $\epsilon : A \to k$. The cohomology of $\mathrm{Q}A$ is denoted $\pi(A)$, or $\pi^n(A) = \mathrm{H}^n(\mathrm{Q}A)$ for $n \geq 1$.

Lemma 1.1. *If A is a minimal k-cdga, then $d(A^+) \subseteq (A^+ \cdot A^+)$. Hence, $\mathrm{Q}^n A \cong \pi^n A$ for all n.*

Proof. It suffices to prove that $\mathrm{Q}^i X(n) = 0$ for $i > n \geq 1$. Now, $\mathrm{Q}^i X(n,0) = X(n-1) = 0$ for $i > n$ by definition, since the generators of $X(n-1)$ have degree at most n. If $\mathrm{Q}^i X(n, q-1) = 0$ for $i > 0$ and some $q \geq 1$, then $d(x) \in (X(n, q-1)^+ \cdot X(n, q-1)^+)$ for $x \in X(n, q)$. Hence, by induction and minimality, $\mathrm{Q}^i X(n) = 0$ for $i > n$. □

We will say that the differential is **decomposable** when $d(A^+) \subseteq A^+ \cdot A^+$. The converse of the last lemma is nearly true. We just need to add the condition that A be 1-connected.

Lemma 1.2. *Suppose that A is a strictly 1-connected k-cdga such that A is free as a graded-commutative k-algebra and $d(A^+) \subseteq (A^+ \cdot A^+)$. Then, A is minimal.*

Proof. The hypothesis in fact implies that $X(n, 1) = X(n)$ for $n \geq 1$. This is trivial for $n = 1$ as $X(1) = X(0) = k$. For $n \geq 2$, the fact that the differential is decomposable means that for degree reasons, if $x \in A^n$, then $d(x)$ is a sum of products of degrees in the range $[2, \ldots, n-1]$. □

Example 1.3. Note that the k-cdga $A = \Lambda_1(x, y) \otimes_d \Lambda_1(z)$ with $d(z) = xy$ from last time has a decomposable differential and is connected. However, it is not 1-connected, so the lemma does not apply to A. We saw last time that A is not minimal.

2. Why do we like characteristic zero?

Let S^n be a sphere with $n > 0$ odd. Let's try to build a small cdga which is free as a graded-commutative $\mathbb{Z}$-algebra with cohomology isomorphic to $\mathrm{H}^*(S^n, \mathbb{Z})$. Here by small I mean minimal, which definition makes just as much sense over a commutative ring R as it does over a field.

Remark 2.1. It turns out that it is impossible to find such an A such that there is a quasi-isomorphism $A \simeq \mathrm{C}^*(S^n, \mathbb{Z})$. But, this is not so important for us below.

The free graded-commutative $\mathbb{Z}$-algebra on an element in degree n is $\mathbb{Z}[x]/(2x^2)$. Call this $\mathbb{Z}$-cdga $A(n)$. (In the lecture there was a long digression on adjoint functors and the notion of a free object. See any book on category theory for details.) In particular, the elements x^m are non-zero for all $n \geq 0$, but $2x^m = 0$ for $m \geq 2$. This is going to force us to introduce infinitely many generators to even build the algebra in some small way. Now, let's introduce a new element z in degree $2n-1$ such that $d(z) = x^2$ and call the resulting cdga $A(2n-1)$. The new underlying graded-commutative algebra is $\mathbb{Z}\langle x, y\rangle/(2x^2, 2y^2, xy + yx)$. The Leibniz rule says that $d(x^n z) = x^{n+1}$. However, $d(2z) = 0$. So, we have killed a lot of the 'bad' cohomology of $A(n)$, but we've introduced new cohomology in degree $2n-1$. So, we will have to adjoin an element to kill this and so on.

Date: 26 January 2016.

Exercise 2.2. Check that the process described above does not terminate.

In other words, one cannot build *in finitely many steps* a strictly connected $\mathbb{Z}$-cdga A that is graded-free as an algebra and has the cohomology of the n-sphere for n odd.

Over $\mathbb{Q}$ we could find such an algebra on one generator, namely $\Lambda_n^{\mathbb{Q}}(x)$.

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