Rational Homotopy Theory - Lecture 1

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1. Differential graded algebras

Let k be a commutative ring. A **differential graded algebra** (or **dga** for short) is a $\mathbb{Z}$-graded k-algebra $A_\bullet$ together with a differential $d : A_\bullet \to A_{\bullet-1}$ satisfying the Leibniz rule: for elements $x \in A_m$ and $y \in A_n$ one has

$$d(xy) = d(x)y + (-1)^m x d(y).$$

Note that this is a **homological** dga. We will also make use of cohomological dgas $A^\bullet$, where the differential increases the degree by 1. The Leibniz rule remains the same. We will often write simply A for $A_\bullet$ or $A^\bullet$.

A differential graded algebra is **commutative** (a **cdga**) if for $x \in A_m$ and $y \in A_n$ we have

$$xy = (-1)^{mn} yx.$$

In other words, a dga is a cdga if it is graded-commutative as a graded ring.

Remark 1.1. Some authors require that in addition $x^2 = 0$ for all homogeneous elements of odd degree x. If 2 is invertible in k this reduces to our definition. Since we will be working typically with $k = \mathbb{Q}$, the distinction is unimportant for our purposes.

Exercise 1.2. Prove that if $A_\bullet$ is a dga (resp. a cdga), then there is a natural induced graded k-algebra structure (resp. graded-commutative k-algebra structure) on $\mathrm{H}_*(A)$.

Example 1.3. Any ordinary graded k-algebra (resp. graded-commutative k-algebra) can be viewed as a dga (resp. cdga) with trivial differential $d = 0$. In particular, ordinary k-algebras (resp. commutative k-algebras) can be viewed as dgas (resp. cdgas) concentrated in degree 0.

Example 1.4. The cochains $\mathrm{C}^\bullet(X, k)$ on a space X with coefficients in k form a dga. Recall that the m-chains on X is the free k-module $\mathrm{C}_m(X, k)$ generated by continuous maps $\Delta^m_{\mathrm{top}} \to X$, where Δ^m_{top} is the **standard m-simplex**, the closed subset of $\mathbb{R}^{n+1}$ of points $(x_0, \ldots, x_n)$ satisfying $x_i \geq 0$ for $0 \leq i \leq n$ and $\sum_{i=0}^n x_i = 1$. The cochains in degree m are the k-linear dual of $\mathrm{C}_m(X, k)$. That is, $\mathrm{C}^m(X, k) = \mathrm{Hom}_k(\mathrm{C}_m(X, k), k)$. The differential is the alternting sum of the maps induced by the inclusion of faces in the standard simplex. There is a multiplication on $\mathrm{C}^\bullet(X, k)$ that makes it into a dga: the cup product $\cup$. If $\alpha \in \mathrm{C}^m(X, k)$ and $\beta \in \mathrm{C}^n(X, k)$, then the value of $\alpha \cup \beta$ on an $(m+n)$-chain $v : \Delta^{m+n}_{\mathrm{top}} \to X$ is

$$\alpha([v_0, \ldots, v_m])\beta([v_m, \ldots, v_{m+n}]),$$

where $[v_0, \ldots, v_m]$ denotes the restriction of v to the subsimplex generated by the first $m+1$ vertices of $\Delta^{m+n}_{\mathrm{top}}$ and similarly for $[v_m, \ldots, v_{m+n}]$. Recall that $\mathrm{H}^*(\mathrm{C}^\bullet(X, k))$ is the singular cohomology of X with coefficients in k. It is a graded-commutative k-algebra.

Exercise 1.5. Prove that $\mathrm{C}^\bullet(X, k)$ is a dga. Show that in general $\mathrm{C}^\bullet(X, k)$ is *not* a cdga despite the fact recalled from algebraic topology that $\mathrm{H}^*(X, k)$ is graded-commutative.

Example 1.6. If M is a k-module, let $P_\bullet \to k$ be a projective resolution. Composition induces a dga structure on $\mathrm{Hom}_k(P_\bullet, P_\bullet)$ which computes the Ext-algebra $\mathrm{Ext}^*_k(M, M)$. See Weibel [2, Section 1.2] for the totalization of double complexes.

The next two examples are particularly important classes of cdgas.

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Example 1.7. If X is a real manifold, $\mathrm{A}^\bullet_{\mathrm{dR}}(X)$ is the de Rham complex of M, which has a cdga structure via the wedge product and the exterior derivative. The de Rham theorem says that $\mathrm{H}^*(\mathrm{A}_{\mathrm{dR}}(X)) \cong \mathrm{H}^*(X, \mathbb{R})$.

Example 1.8. Let $X = \operatorname{Spec} R$ be a smooth affine k-scheme, and let $\Omega^1_{X/k}$ denote the coherent sheaf of Kähler differentials on X. The exterior algebra $\wedge_R \Omega^1_{X/k}$ where $\Omega^1_{X/k}$ is placed in degree 1 is called the **algebraic de Rham complex**. it is a cdga with differential induced by $r \mapsto dr$. We usually write $\mathrm{H}^*_{\mathrm{dR}}(X/k)$ for the cohomology.

A morphism $f : A \to B$ of dgas over k is a **quasi-isomorphism** if $\mathrm{H}_*(f) : \mathrm{H}_*(A) \to \mathrm{H}_*(B)$ is an isomorphism. Two dgas over k are **quasi-isomorphic** if there is a sequence of zig-zags $A \leftarrow C(0) \to C(1) \leftarrow C(2) \to \cdots \leftarrow C(m) \to B$ where each $C(i)$ is a dga over k and each map is a quasi-isomorphism of dgas over k. We make the same definition for cdgas.

Warning 1.9. Two cdgas may be quasi-isomorphic as dgas but not as cdgas. This happens when one can find intermediary dgas $C(i)$ and quasi-isomorphisms but not intermediate cdgas. It is possible to give an example of non-quasi-isomorphic cdgas with isomorphic homology rings. Any non-formal cdga will suffice, and we find one at the end of the lecture.

Definition 1.10. A dga (resp. a cdga) A is formal if it is quasi-isomorphic as a dga (resp. as a cdga) to $\mathrm{H}_*(A)$ equipped with the trivial differential.

The following is one of the motivating problems of the subject, and is motivated by the fact that one can replace $\mathrm{C}^\bullet(X, \mathbb{R})$ with $\mathrm{A}^\bullet_{\mathrm{dR}}(X)$ when X is a manifold, the latter being a cdga.

Problem 1.11 (Thom)**.** *Is it possible to assign naturally to every space X a cdga $\mathrm{A}^\bullet(X, \mathbb{Q})$ over $\mathbb{Q}$ that is quasi-isomorphic as a dga to $\mathrm{C}^\bullet(X, \mathbb{Q})$?*

Rational homotopy theory solves this problem. Note that without $\mathbb{Q}$ coefficients, it is not possible to solve the problem. If $\tilde{\mathrm{H}}^*(X, \mathbb{F}_\ell)$ is non-trivial for some prime ℓ, then $\mathrm{C}^\bullet(X, \mathbb{Z})$ is *never* quasi-isomorphic to a cdga! One proves this fact using the existence of Steenrod operations.

2. Formality and Kähler groups

A group G is Kähler if it is isomorphic to $\pi_1(X)$ where X is a compact Kähler manifold. All such groups are finitely presented.

Question 2.1. Which finitely presented groups are Kähler?

Note that all finitely presented groups are fundamental groups of 2-dimensional CW complexes or simplicial complexes, 4-dimensional almost-complex symplectic manifolds, or real 6-dimensional complex symplectic manifolds.

Example 2.2.
(1) Serre proved that all finite groups are Kähler. It is enough to show that each finite symmetric group S_n is Kähler. Indeed, finite covers of Kähler manifolds admit Kähler metrics, so the class of Kähler groups is closed under passage to finite index subgroups. Serre takes a generic high-codimension linear section of the quotient $(\mathbb{CP}^2)^n/S_n$.
(2) The groups $\mathbb{Z}^{2n}$ are Kähler, as they are the fundamental groups of complex tori.
(3) The groups $\mathbb{Z}^{2n+1}$ are *not* Kähler. Indeed, for any space X, the map $X \to \mathrm{B}\pi_1(X)$ induces an isomorphism $\mathrm{H}^1(\mathrm{B}\pi_1(X), \mathbb{Z}) \to \mathrm{H}^1(X, \mathbb{Z})$ and also an injection in degree 2. Hodge theory says that $\mathrm{H}^2(X, \mathbb{Z})$ must have even rank. On the other hand, $\mathrm{H}^1(\mathrm{B}\mathbb{Z}^{2n+1}, \mathbb{Z}) \cong \mathbb{Z}^{2n+1}$.

Rational homotopy theory leads to some new restrictions on Kähler groups, which are finer than those given by just the Hodge decomposition.

Theorem 2.3 (Deligne-Griffiths-Morgan-Sullivan [1])**.** *If X is a compact Kähler manifold, then $\mathrm{A}^\bullet_{\mathrm{dR}}(X)$ is formal.*

To use this, we introduce the notion of Massey triple products. Let $A^\bullet$ be a cohomological cdga, and let a, b, c be homogeneous elements of $\mathrm{H}^*(A)$ such that $ab = 0$ and $bc = 0$ in $\mathrm{H}^*(A)$. Pick representatives α of a, β of b, and γ of c. Note that there must exist elements η and ρ of $A^\bullet$ such that

$$d\eta = \alpha\beta$$
$$d\rho = \beta\gamma$$

by the definition of cohomology. Let $\langle a, b, c\rangle$ denote the class of

$$\eta\gamma + (-1)^{|a|+1}\alpha\rho$$

in $\mathrm{H}^{|a|+|b|+|c|-1}(A)$.

Exercise 2.4. Show that $\langle a, b, c\rangle$ is well-defined in $\mathrm{H}^*(A)/(a, c)$, where (a, c) is the ideal generated by a and c.

The resulting class is called a **Massey triple product**. We will see that it is a finer invariant of a cdga.

Exercise 2.5. Show that if A is formal, then all Massey products vanish. To do this you will also have to show that Massey triple products behave well under maps of cdgas.

Recall that if X is a connected space, then the map $\mathrm{H}^*(\mathrm{B}\pi_1(X), \mathbb{Q}) \to \mathrm{H}^*(X, \mathbb{Q})$ is injective up to degree 2. It follows that if a Massey triple product $\langle a, b, c\rangle$ is non-zero in group cohomology where a, b, c have degree 1, then it is non-zero in the cohomology of X. We will use this and the result of DGMS above to give another example of a non-Kähler group.

Let N^3 denote the group of matrices

$$\begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$$

where $x, y, z \in \mathbb{R}$. Let $\Gamma = \mathrm{N}^3 \cap \mathrm{SL}_3(\mathbb{Z})$ inside $\mathrm{SL}_3(\mathbb{R})$. The quotient $X = \mathrm{N}^3/\Gamma$ is a closed orientable 3-manifold called the Heisenberg manifold.

Exercise 2.6. Projection onto the x, z coordinates induces a central extension

$$0 \to \mathbb{Z} \to \Gamma \to \mathbb{Z}^2 \to 0.$$

Show that the Heisenberg manifold X is the total space of an S^1-bundle (a circle bundle) over the torus T^2.

Using rational homotoy theory and the theory of Hirsch extensions, it turns out that $\mathrm{A}^\bullet_{\mathrm{dR}}(X, \mathbb{R})$ can be computed using a so-called minimal model $M^\bullet$ which in this case takes the form $M^\bullet = \lambda_1(a, b, c)$, the exterior algebra generated by elements a, b, c in degree 1 with differentials $d(a) = d(b) = 0$ and $d(c) = ab$.

Exercise 2.7. Compute the cohomology ring of $M^\bullet$ and show that the Massey triple product $\langle a, b, c\rangle$ is non-zero in $\mathrm{H}^2(X, \mathbb{R})$.

As a consequence of what we have said above, the theorem of DGMS implies that Γ is not a Kähler group. Note also that $M^\bullet$ cannot be formal.

References

[1] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975), no. 3, 245–274.

[2] C. A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994.