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Lecture 7.

28 January 2015.

<u>Fibrant replacement.</u> Let $f: X \to Y$. We can factor f as

$$X \xrightarrow{c} Pf \xrightarrow{p} Y,$$

where c is a homotopy equivalence and p is a fibration.

<u>proof.</u> $Pf = \{ (x, \phi) \in X \times PY \mid f(x) = ev_0(\phi) \}$. The map c includes X by the constant paths, while $p = ev_1$. By contracting paths, we see that Pf deformation retracts onto X. Hence, c is a homotopy equivalence.

One checks that the evaluation map is continuous. In fact, continuity of evaluation maps is built in to the compact-open topology. See Hatcher, Appendix A.

Now, consider a diagram

$$\begin{tikzcd} A \times \{0\} \arrow[r, "g_0"] \arrow[d] & Pf \arrow[d, "ev_1"] \\ A \times I' \arrow[r, "h"'] & Y. \end{tikzcd}$$

We can write $g_0(a) = (\chi(a), \phi(a)) \in X \times PY$. Set

$$g_t(a) = (\chi(a), \phi(a) \cdot h_{[0,t]}(a)),$$

which makes sense as $\phi(a)(1) = h(a)(0)$.

<u>Remark.</u> The map $X \to *$ is a fibration for any space X.

<u>Proposition.</u> Let $p : X \to Y$ be a fibration. If $b, c \in Y$ are in the same path-component, then X_b is homotopy equivalent to X_c.

<u>proof.</u> Let $\phi : I' \to Y$ be a path. It suffices to show that $p^{-1}(\phi(0)) \simeq p^{-1}(\phi(1))$. Consider the lifting problem

$$\begin{tikzcd}
X_{\phi(0)} \times \{0\} \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
X_{\phi(0)} \times I' \arrow[r, "h"] \arrow[ur, dashed, "g"] & Y
\end{tikzcd},$$

where $h(x,t) = \phi(t)$. Since p is a fibration, a lift g exists, and defines a map $X_{\phi(0)} \to X_{\phi(1)}$ by restriction to $X_{\phi(0)} \times \{1\}$ and commutativity.

Suppose $\phi \simeq \gamma$ rel $\partial I'$. We get two maps $X_{\phi(0)} \to X_{\phi(1)}$. It turns out that they're homotopic. We pick

$$\begin{tikzcd}
X_{\phi(0)} \times I \times \{0\} \cup X_{\phi(0)} \times \partial I' \times I' \arrow[r] \arrow[d] & X \arrow[d, "p"] \\
X_{\phi(0)} \times I' \times I' \arrow[r, "k"'] \arrow[ur, dashed, "m"] & Y
\end{tikzcd},$$

inclusion of $X_{\phi(0)}$ on $X_{\phi(0)} \times I' \times \{0\}$, and the two lifts as shown for ϕ, γ on $X_{\phi(0)} \times \partial I' \times I'$.

where $k(x,s,t) = j(s,t)$, a homotopy from ϕ to γ rel $\partial I'$.

Note that $(I \times I,\ I \times \{0\} \cup \partial I \times I) \cong (I \times I,\ I \times \{0\})$.

Hence fibrations also satisfy this lifting property, and a map m as in the diagram exists. $m(x,1,t)$ is the desired homotopy. In particular, $\phi \circ \phi^{-1} \simeq \mathrm{id}_{X_{\phi(1)}}$ and $\phi^{-1} \circ \phi \simeq \mathrm{id}_{X_{\phi(0)}}$.

If Y is a space, $Top_{/Y}$ is the category of spaces with a fixed map to Y, as $f: X \to Y$. In $Top_{/Y}$, the notion of homotopy is fiber-wise. For example, two spaces $f: X \to Y$ and $g: Z \to Y$ over Y are <u>homotopy equivalent over Y</u>, or fiber-wise homotopy equivalent, if there are maps $a: X \to Y$ and $b: Y \to Z$ in $Top_{/Y}$ s.t. $a \circ b \simeq id_Y$ and $b \circ a \simeq id_Z$ <u>in</u> $Top_{/Y}$. So, one has various commutative diagrams

$$\begin{tikzcd} X \arrow[rr, "a"] \arrow[dr, "f"'] & & Z \arrow[dl, "g"] \\ & Y & \end{tikzcd} , \quad \begin{tikzcd} Z \arrow[rr, "b"] \arrow[dr, "g"'] & & X \arrow[dl, "f"] \\ & Y & \end{tikzcd} , \quad \begin{tikzcd} X \times \partial I^1 \arrow[d] \arrow[dr, "id_X \sqcup b \circ a"] & \\ X \times I^1 \arrow[r] \arrow[d] & X \arrow[dl, "f"] \\ Y & \end{tikzcd} .$$

<u>Proposition</u>. If Y is locally contractible, then any fibration $X \xrightarrow{p} Y$ is locally fiber homotopy equivalent to a product fibration.

<u>proof</u>. We can assume Y is contractible, so that $id_Y \simeq * \in Y$. It suffices then to show that if $A \times I^1 \xrightarrow{h} Y$, then $h_0^* p \simeq h_1^* p$ over A. The idea is to follow the argument in the previous proposition:

$$\begin{tikzcd} A \times \{c\} \arrow[r, "\phi(c)\text{ fiber}"] \arrow[d] & h^* X \arrow[d] \\ A \times I^1 \arrow[r, "{(id, \phi)}"'] \arrow[ur, dashed] & A \times I^1 \end{tikzcd} .$$

This automatically preserves fibers.

<u>Motto</u>. Fibrations are homotopy fiber bundles.