548.
Lecture 6.
26 January 2015.

<u>Definition</u>. A <u>pointed pair</u> is the data $(X, A, *)$ of a space X, a subspace $A \subseteq X$, and a base point $* \in A \subseteq X$.
One can consider maps and homotopy classes of maps

$$(X, A, *) \longrightarrow (Y, B, *)$$

in the usual way. Maps are pointed maps $f: (X,*) \longrightarrow (Y,*)$ such that $f(A) \subseteq B$. ~~Two such maps are homotopic rel A if there is a map $h: I \times X \to Y$ such that $h(t, a) = f(a) = g(a)$ for $t \in [0,1]$, $a \in A$.~~

<u>Warning</u>. Homotopies of maps of pairs are not only the homotopies rel A.

<u>Definition</u>. For $n \geq 1$, define

$$\pi_n(X, A, *) = [(D^n, S^{n-1}, s), (X, A, *)]_*,$$

the <u>relative homotopy groups</u> of the pair.
For $n \geq 2$ one has a bifurcation map $D^n \longrightarrow D^n \vee D^n$, making them into groups, abelian if $n \geq 3$.

<u>Ex</u>. $\pi_n(X, *, *) = \pi_n(X, *)$ for $n \geq 1$.

<u>Alternate definition</u>. If $f: X \longrightarrow Y$ is a continuous pointed map, the <u>pointed mapping path space</u> Pf is

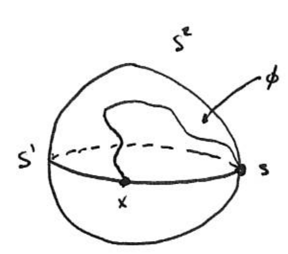
$$P_*f = \{ (x, \phi) \in X \times PY \mid f(x) = ev_0(\phi) \text{ and } ev_1(\phi) = * \}.$$

Here $PY = \mathrm{map}(I, Y)$, the space of unpointed maps. Note that P_*Y = the mapping path space of inclusion of the base point.

We can define Pf as the fiber product

$$\begin{tikzcd} P_*f \arrow[r] \arrow[d] & P_*Y \arrow[d, "ev_0"] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

When $i: A \hookrightarrow X$ is a subspace, P_*i consists of maps from the basepoint of X to a point of A, but where the map can travel in X.



$(x, \phi) \in Pi$,
$i: S^1 \hookrightarrow S^2$.

The nth relative homotopy group is

$$\pi_n(X, A, *) = \pi_{n-1}(Pf),$$

where Pf is pointed by the constant path at the basepoint, $f : A \hookrightarrow X$. This proves the claim above about group structures.

<u>Ex.</u> Think about $n = 1$. A map $(D^1, S^0, *) \to (X, A, *)$ is exactly a path in X from $*$ to a point in A.

<u>Thm.</u> The relative homotopy groups of the pair $(X, A, *)$ fit into a long exact sequence

$$\cdots \to \pi_n(X, A, *) \to \pi_{n-1}(A, *) \to \pi_{n-1}(X, *) \to \pi_{n-1}(X, A, *) \to \cdots$$

$$\to \cdots \pi_1(X, *) \to \pi_1(X, A, *) \to \pi_0(A, *) \to \pi_0(X, *).$$

Exactness at the bottom means that

(i) the fiber of $\pi_0(A, *) \to \pi_0(X, *)$ is the image of $\pi_1(X, A, *)$,

(ii) the fiber of $\pi_1(X, A, *) \to \pi_0(A, *)$ is the coset $\pi_1(X, *)/\pi_1(A, *)$, and

(iii) the fiber of $\pi_1(X, *) \to \pi_1(X, A, *)$ is the subgroup $\pi_1(A, *)/\pi_2(X, A, *)$.

The proof will take a little while, and we will prove something a bit more general.

Definition. A map $f: X \to Y$ is a (*Hurewicz*) *fibration* if it satisfies HLP with respect to all spaces A:

$$\begin{tikzcd} A \times \{0\} \arrow[r] \arrow[d] & X \arrow[d] \\ A \times I \arrow[r] \arrow[ur, dashed, "\exists"] & Y \end{tikzcd} \qquad \text{(HLP)}.$$

Remark. In the pointed case, being a fibration only depends on the connected component of Y containing the basepoint.

There is an obvious pointed version of this, giving *pointed* (*Hurewicz*) *fibrations*.

Ex. If $\tilde{X} \to X$ is a covering space, it is a fibration. For these, the lifts are unique.

Definition. A map $f: X \to Y$ is a *Serre fibration* if it satisfies HLP with respect to disks D^n.

Definition. A *fiber bundle* with fiber F consists of a map $E \xrightarrow{p} B$ such that each point $b \in B$ has an open neighborhood $b \in U \subseteq B$ such that $p^{-1}(U) \cong U \times F$ so that

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "p"'] & & U \times F \arrow[dl] \\ & U & \end{tikzcd}$$

commutes. The space E is the *total space*, while B is the *base space*.

Ex. If F is discrete, a fiber bundle with fiber F is a covering space, and the converse is true if B is path-connected.

Ex. If M^k is a smooth k-dimensional manifold, $TM \to M$ is a fiber bundle with fiber $\mathbb{R}^k$. This is an example of a vector bundle. Are all fiber bundles with fiber $\mathbb{R}^k$ vector bundles?

Proposition. If $p: E \to B$ is a fiber bundle with fiber F, then p is a Serre fibration.

proof. Instead of D^n, we'll use I^n, and induct on n. Consider

$$\begin{tikzcd} I^n \times \{0\} \arrow[r, "g_0"] \arrow[d] & E \arrow[d] \\ I^n \times I^1 \arrow[r, "h"'] \arrow[ur, dashed, "g"] & B, \end{tikzcd}$$

Take a cover $\{U_i\}$ of B s.t. $p^{-1}(U_i) \cong U_i \times F$ over U_i. Using compactness, divide I^n into finitely many subcubes C_α and I^1 into finitely many intervals I_β s.t. h maps $C_\alpha \times I_\beta$ into a single open $U_{\alpha,\beta}$.

Inductively, assume we have constructed $g(x,t)$ for $x \in \bigcup_\alpha \partial C_\alpha$. To extend g to $C_\alpha \times I^1$, we can proceed by lifting to each $C_\alpha \times I_\beta$ one at a time. But, here we have

Just argue first that no subdivision is required.

$$\begin{tikzcd} C_\alpha \times \{0\} \arrow[r, "g_0"] \arrow[d] & p^{-1}(U) \cong U \times F \arrow[d] \\ C_\alpha \times I_\beta \arrow[r] \arrow[ur, dashed, "g"] & U \subseteq B \end{tikzcd}$$

and we can define $g(x,t)$ by $(h(x,t), g_0(x))$.

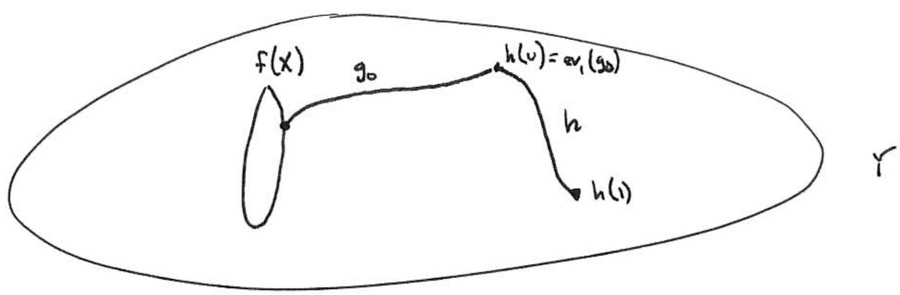
Picture. $n=1$:

$$\begin{tikzcd} I^1 \times \{0\} \arrow[r, "g_0"] \arrow[d] & B \times F \arrow[d] \\ I^1 \times I^1 \arrow[r, "h"'] \arrow[ur, dashed] & B \end{tikzcd}$$

$$g(x,t) = (h(x,t), g_0(x)).$$

Remark. $B \times F \longrightarrow F$ is a Hurewicz fibration. If B is paracompact, fiber bundles with base B are Hurewicz fibrations.

When $A = *$, this might look like



<u>Definition</u>. The <u>homotopy fiber</u> of a map $f : X \to Y$ at a basepoint $y \in Y$ is the pullback

$$\begin{array}{ccc} P_y f = F & \longrightarrow & Pf \\ \downarrow & & \downarrow{\scriptstyle ev_1} \\ * & \xrightarrow[y]{} & Y \end{array}$$

This the homotopy fiber product $* \times^h_Y X$.