<u>Lemma</u>. Let X, Y be based spaces. Then,

$$[X,-]_* : Top_+ \longrightarrow Sets_*$$

$$[-,Y]_* : Top_+^{op} \longrightarrow Sets_*$$

are functors.

<u>proof</u>. Let's prove the first. If $Z \in Top_+$,

$$[X,Z]_* = \mathrm{Hom}_{Top_+}(X,Z) \,/\, \text{homotopy}.$$

If $f \in Z \longrightarrow W$, $g \in \mathrm{Hom}_{Top_+}(X,Z)$,

$$f \circ g \in \mathrm{Hom}_{Top_+}(X,Z).$$

We get

$$\mathrm{Hom}_{Top_+}(X,Z) \longrightarrow \mathrm{Hom}_{Top_+}(X,W) \longrightarrow [X,W]_*.$$

We need to show that if $g \sim h \in \mathrm{Hom}_{Top_+}(X,Z)$, then $f\circ g \sim f \circ h \in \mathrm{Hom}_{Top_+}(X,W)$. If $j : I_+ \wedge X \longrightarrow Z$ is such a homotopy, then $f \circ j$ is a homotopy from $f\circ g$ to $f \circ h$. The conditions for being a functor are immediate. The assignment presevers identities and composition.

<u>Lemma</u>. If $f \sim g : Y \longrightarrow Z$, then the two maps

$$[X,Y]_* \underset{[X,g]_*}{\overset{[X,f]_*}{\rightrightarrows}} [X,Z]_*$$

agree.

These in fact induce natural isomorphisms of functors $[-,Y]_* \longrightarrow [-,Z]_*$, which thus agree.

Lemma. If $Y \xrightarrow{f} Z$ is a pointed homotopy equivalence, then

$$[X,Y]_* \longrightarrow [X,Z]_*$$

define this: pointed homotopy inverses.

is a bijection for any pointed X.

proof. Let g be a homotopy inverse of f. Hence, $f \circ g \sim id_Z$ and $g \circ f \sim id_Y$. It follows that the two compositions

$$[X,Y]_* \xrightarrow{[X,f]_*} [X,Z]_* \xrightarrow{[X,g]_*} [X,Y]_*,$$

$$[X,Z]_* \longrightarrow [X,Y]_* \longrightarrow [X,Z]_*$$

are the identity.

Example. $\mathbb{R}^n$ is contractible: the inclusion $\vec{0} \longrightarrow \mathbb{R}^n$ is a homotopy equivalence. Indeed

$$h : I \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$h(t, \vec{x}) = t\vec{x}$$

is a homotopy from the point to the identity.

Since $\pi_n(*) = *$ for all $n \geq 0$, we see from the lemma that $\pi_n(\mathbb{R}^m) = *$ for all $n, m \geq 0$.

<u>Function spaces</u>. If X, Y are pointed topological spaces,

$\mathrm{map}_*(X,Y)$ is the set of all continuous pointed maps from X to Y with the topology generated by sets $N_{K,U} = \{f \mid f(K) \subset U\}$ $K \subseteq X$ compact, $U \subseteq Y$ open. This is the <u>compact-open</u> topology.

We call $\mathrm{map}_*(X,Y)$ the function space or the mapping space.

<u>Ex</u>. $\Omega X = \mathrm{map}_*(S^1, X)$, the loop space of X.

<u>Proposition</u>. Let X, Y, Z be pointed spaces, where X, Z are Hausdorff and Z is additionally locally compact. Then, there is a natural bijection

Loc. cpt. means that each point has an open nbd. whose closure is cpt.

$$a : [Z \wedge X, Y]_* \xrightarrow{\cong} [X, \mathrm{map}_*(Z,Y)]_*,$$

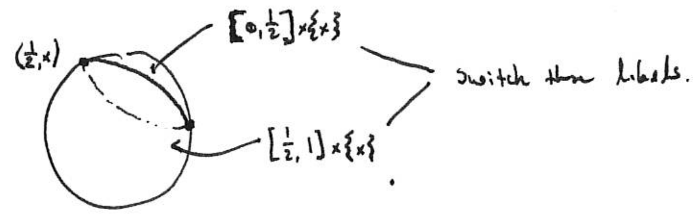
defined by sending $f : \cancel{Z} \wedge X \to Y$ to the function $a(f)$ defined by
$$[(a(f))(x)](z) = f(z,x).$$

<u>proof</u>. This will be a homework exercise.

<u>Corollary</u>. $\pi_1(Y) = \pi_0(\Omega Y)$.

Proposition. $S^1 \wedge S^n \cong S^{n+1}$.

proof. Here's an $n=1$ picture. $S^1 \wedge S^n$ is $I^1 \times S^n / I^1 \times \{*\} \cup \{0,1\} \times S^n$.



We see that we're tracing out the 2-sphere. Generally, let

$S^{n+1} \subseteq \mathbb{R}^{n+2}$,

$S^n \subseteq S^{n+1}$ where $x_{n+2} = 0$,

$D^{n+1} \subseteq \mathbb{R}^{n+2}$ as $\{ x \in \mathbb{R}^{n+2} \mid |x| \leq 1 \text{ and } x_{n+2} = 0 \}$,

H_+^{n+1} = upper hemisphere of S^{n+1}

H_-^{n+1} = lower hemisphere of S^{n+1}.

$$P_+ : (D^{n+1}, S^n) \cong (H_+^{n+1}, S^n),$$

$$P_- : (D^{n+1}, S^n) \cong (H_-^{n+1}, S^n)$$

homeomorphisms, inverse to projecting onto the $x_{n+2} = 0$ plane.

Define $h : I^1 \times S^n \longrightarrow S^{n+1}$ by

$$h(t,x) = \begin{cases} P_-(2tx + (1-2t)s_0) & t \in [0, \frac{1}{2}] \\ P_+(2(1-t)x + (2t-1)s_0) & t \in [\frac{1}{2}, 0]. \end{cases}$$

$s_0 = (1, 0, \ldots, 0)$, the base point. This gives a continuous bijection $S^1 \wedge S^n \to S^{n+1}$, hence a homeomorphism as they're compact, Hausdorff.