Thm (Hurewicz). If X is $(n-1)$-connected for $n\geq 2$, then $\tilde{H}_i(X)=0$ for $i\leq n-1$ and $\tilde{H}_n(X)\cong\pi_n(X)$. If (X,A) is $(n-1)$-connected, $n\geq 2$, and A is simply-connected and nonempty, then $H_i(X,A)=0$ $i<n$, and $\pi_n(X,A)\cong H_n(X,A)$.

proof. Can replace (X,A) by a whe CW pair. Then, $\underset{(i\leq n)}{\pi_i(X,A)\cong\pi_i(X/A)}$ ← BME and $\underset{(\forall i)}{\tilde{H}_i(X,A)\cong\tilde{H}_i(X/A)}$. So, reduce to absolute case.

So, assume X is an $(n-1)$-connected CW complex, $n\geq 2$. Can assume $X^{n-1}=*$, and we may as well assume $X^{n+1}=X$. So,

$$X=\left(\vee S^n_\alpha\right)\cup_\beta e^{n+1}_\beta$$

for some $\phi_\beta: S^n\to\vee S^n_\alpha$. ← pointed. Hence,

$$\pi_n(X)=\operatorname{coker}\left(\bigoplus_\beta\mathbb{Z}\xrightarrow{\phi_\beta}\bigoplus_\alpha\mathbb{Z}\right).$$

But, this is also $\tilde{H}_n(X)$ by cellular homology:

$$\begin{array}{ccccccc} H_{n+1}(X^{n+1},X^n) & \to & H_n(X^n,X^{n-1}) & \to & \tilde{H}_n(X) & \to & H_n(X^{n+1},X^n) \\ & & \| & & & & \| \\ & & H_n(X^{n+1},X^{n-1}) & & & & 0 \end{array}.$$

Cor. If $f:X\to Y$ is a map between simply connected CW complex s.t. $f_*:H_*(X)\cong H_*(Y)$, then f is a homotopy equivalence.

proof. Can assume $X\hookrightarrow Y$ is a CW pair. By hypothesis, $\pi_1(Y,X)=0$. First non-zero $\pi_n(Y,X)$ is first non-zero $H_n(Y,X)$. Hence, f is a whe. Hence, by Whitehead, a he.

Cor. Let X be a $K(A,n)$ for $n \geq 1$. Then,

$$\tilde{H}_n(X,\mathbb{Z}) \cong A,$$

$$H^n(X,\mathbb{Z}) \cong \mathrm{Hom}(A,\mathbb{Z}).$$

proof. This follows immediately from Hurewicz.

Co-building blocks feat. PT.

Let X be a topological space, path connected for simplicity.

Consider the process: kill the homotopy groups of X above degree i.

$$\begin{tikzcd}
 & \vdots \arrow[d] \\
 & X_2 \arrow[d] \\
 & X_1 \arrow[d] \\
X \arrow[r] \arrow[ur] \arrow[uur] \arrow[uuur] & *
\end{tikzcd}$$

So,
$$\pi_k X_i \cong \begin{cases} \pi_k X & k \leq i \\ 0 & k > i, \end{cases} \quad \text{and} \quad \pi_k X \to \pi_k X_i$$
is an iso for $0 \leq k \leq i$.

Q. What is the homotopy fiber of $X_i \to X_{i-1}$?

The tower $\{X_i\}$ is called the Postnikov tower of X. We see that X is co-built from Eilenberg-MacLane spaces.

Thm. $[X, K(A,n)]_* \cong H^n(X,A)$.

sketch. $H^n(K(A,n),A) \simeq \mathrm{Hom}(H_n(A),A)$

$$\begin{array}{c} \mathrm{Hom}(H_n(A), A) \\ \| \\ \pi_n K(A,n). \end{array}$$

Set $\alpha \in H^n(K(A,n),A)$ the inverse of Hurewicz.
Given $X \xrightarrow{f} K(A,n)$, get $f^*(\alpha) \in H^n(X,A)$.

Idea: it's true for $X = S^k$. So, it should be true in general.

Orientability.

$$BSO_n \longrightarrow BO_n \longrightarrow B\mathbb{Z}/2 \simeq K(\mathbb{Z}/2,1)$$

$\uparrow$

fiber of $BSO_n \longrightarrow (BSO_n)_1 \simeq B\mathbb{Z}/2$.

The first Stiefel-Whitney class: $w_1(\xi) \in H^1(X, \mathbb{Z}/2)$

The obstruction to orientability.