Homotopy groups don't satisfy excision:

(1) they behave well wrt limits, because we map out of S^n;

(2) suppose that they did, so we'd have a long exact sequence

$$\cdots \to \pi_{n+1}(S^2) \to \pi_n(S^1) \to \pi_n(D^2_+) \oplus \pi_n(D^2_-) \to \pi_n(S^2) \to \cdots.$$

Of course, $\pi_n(D^2_+) = \pi_n(D^2_-) = 0$ for all n, so

$$\pi_{n+1}(S^2) \cong \pi_n(S^1).$$

So, we would have $\pi_2(S^2) = \mathbb{Z}$ and $\pi_n(S^2) = 0$ for $n \neq 2$.
We'll see (this lecture) that $\pi_3(S^2) \cong \mathbb{Z}$, so excision fails.

The next best thing:

Thm (Blakers–Massey Excision). Suppose $X = A \cup B$, where A, B are subcomplexes of the CW complex with $A \cap B = C \neq \emptyset$. Assume (A, C) is m-connected, (B, C) n-connected. Then,

$$\pi_i(A, C) \to \pi_i(X, B)$$

is an iso for $i < m+n$ and a surjection for $i = m+n$.

Today, we're users.

<u>Thm</u> (Freudenthal Suspension Theorem). Let X be an $(n-1)$-connected space. Then, $\pi_i($[redacted]$X) \to \pi_{i+1}(SX)$ is an iso. for $i < 2n-1$ and a surjection for $i = 2n-1$.

Think about $X = S^n$.

<u>proof.</u> $SX = CX_+ \cup CX_-$, with intersection X.

$$\pi_i(X) \cong \pi_{i+1}(C_+X, X) \to \pi_{i+1}(SX, C_-X) \cong \pi_{i+1}(SX).$$

Now, (C_+X, X) and (CX, X) are both n-connected, so applying BME, we get an iso for

$i+1 < 2n$ $\quad (i < 2n-1)$,

surjection for

$i+1 = 2n$ $\quad (i = 2n-1)$.

<u>Def.</u> A <u>well-pointed</u> space (X, x) has $x \hookrightarrow X$ a cofibration.

<u>Ex.</u> If $x \in X^0$, [redacted] X a CW complex, then (X, x) is well-pointed.

<u>Lemma.</u> For well-pointed spaces X, $SX \to \Sigma X$ is a homotopy equivalence.

So, we get $\pi_i(X) \cong \pi_i(\Sigma X)$ for $i < 2n-1$ as above.

<u>Def.</u> For any space X, define the <u>stable</u> <u>homotopy groups</u> as

$$\pi_n^s(X) = \operatorname*{colim}_{k \geq 0} \pi_{n+k}(\Sigma^k X).$$

<u>Def.</u> $\pi_n^s = \pi_n^s(S^0)$ are called the <u>stable</u> <u>homotopy groups of spheres</u>.

Why stable? For any X, $\Sigma^k X$ is $(k-1)$-connected. So, by Freudenthal, if $n+k < 2k-1$, or $n < k-1$, then

$$\pi_{n+k}(\Sigma^k X) \xrightarrow{\cong} \pi_{n+k+1}(\Sigma^{k+1} X).$$

So, this colimit stabilizes quite quickly. To compute $\pi_n^s(X)$, we can just compute $\pi_{2n+2}(\Sigma^{n+2} X)$.

Ex. $\pi_0^s = \pi_2(S^2)$, $\pi_1^s = \pi_4(S^3)$, $\pi_2^s = \pi_6(S^4), \ldots$

Also, $\pi_n^s(X) \cong \pi_{n+1}^s(\Sigma X)$, so that

$$\pi_n^s = \pi_n^s(S^0) \cong \pi_{n+k}^s(S^k)$$

for all $k \geq 0$.

Remark. It turns out that π_n^s are much easier to compute than $\pi_n(S^k)$. Still impossible.

Proposition. $\pi_n(S^n) \xrightarrow[\deg]{\cong} \mathbb{Z}$, $\pi_0^s \cong \mathbb{Z}$.

proof. $\pi_1(S^1) \to \pi_2(S^2)$ is a surjection by Freudenthal. Hence, $\pi_2(S^2) = \mathbb{Z}/m$ for some m. But, there are degree k maps for all k, so $m=0$. I.e., $\pi_2(S^2) = \mathbb{Z}$. Then,

$$\pi_2(S^2) \xrightarrow[F]{\cong} \pi_3(S^3) \xrightarrow[F]{\cong} \pi_4(S^3) \xrightarrow[F]{\cong} \cdots.$$

Can also get $\pi_1(S^1) \xrightarrow{\cong} \pi_2(S^2)$ from Hopf Fibration $S^1 \to S^3 \to S^2$.

Def. S pointed set,

G group,

A abelian group.

Eilenberg-MacLane spaces.

A $K(A,n)$-space is a pointed top. space X with $\pi_i(X) \cong \begin{cases} A & i=n \\ 0 & \text{otherwise.} \end{cases}$

Similarly for a $K(G,1)$ and $K(S,0)$.

Proposition. Eilenberg-MacLanes exist.

proof. $\coprod_{s \in S} *$ constructs $K(S,0)$.

Q: how to construct $K(G,1)$?

Pick generators α for A as an ab. grp. Set

$$X^n = \bigvee_\alpha S^n_\alpha.$$

So, $\pi_i(X^n) = 0$ for $i<n$, and $\pi_n(X^n) \cong \bigoplus_\alpha \mathbb{Z}$. Attach cells to kill relations. Get X^{n+1} with

$$\pi_i(X^{n+1}) = 0 \quad i<n,$$
$$\pi_n(X^{n+1}) = A.$$

Now, kill off all higher homotopy groups one dim at a time.

Remember: $\pi_i(X^k) \to \pi_i(X)$ is an iso for $i<k$ and a surjection for $i=k$.