

Thm (Whitehead, 1949). Suppose that X, Y are based connected CW-complexes. A weak homotopy equivalence $F: X \rightarrow Y$ is a homotopy equivalence.

Compression Lemma. (X, A) a CW pair,
 (Y, B) a pair with $B \neq \emptyset$. Assume that if $X - A$ has n -cells then $\pi_n(Y, B, \gamma) = 0$ for all choices of γ (meaning (Y, B) n -connected if $n=0$).

Then, every $f: (X, A) \rightarrow (Y, B)$ is homotopic rel A to a map $X \rightarrow B$.

proof. ~~we assume that X^0 maps via f to B~~

Assume that $h: [0, 1 - \frac{1}{2^{k+1}}] \times X \rightarrow Y$ is a homotopy rel A from f to a map $X \rightarrow Y$ that takes X^k to B .

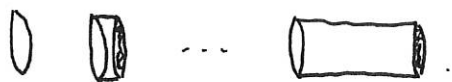
Let $g = h|_{[1 - \frac{1}{2^{k+1}}, 1]}$. Then, for each $(k+1)$ -cell of $X - A$, with attaching map $\phi_\alpha: S^k \rightarrow X$, we get a map

$$(D^{k+1}, S^k) \rightarrow (Y, B),$$

which is necessarily nullhomotopic since $\pi_{k+1}(Y, B, \gamma) = 0$. For each α we get a homotopy rel S^k to $D^{k+1} \rightarrow B$. By definition of X^{k+1} , we get a homotopy from $g|_{X^{k+1}}$ to a map that sends X^{k+1} to B . By HEP for a CW pair, ~~this~~ this homotopy extends to one on all of X . Proceed that along $[1 - \frac{1}{2^{k+1}}, 1 - \frac{1}{2^{k+2}}]$.

Relation of $\pi_n(Y, B)$ to homotopies rel S^{n-1} .

Given $D^n \times I' \longrightarrow Y$ mapping $S^{n-1} \times I$ to B , view $D^n \times I'$ as the cross



Then have the same boundary S^{n-1} , and we get a homotopy rel S^{n-1} .

Repluing a map with a cofibration.

Let ~~$X \xrightarrow{f} Y$~~ be any map, and consider the mapping cylinder (pointed)

$$\begin{array}{ccc}
 X & \xrightarrow{\quad} & Y \\
 (x_0) \downarrow \text{dashed} & & \downarrow \text{dashed} \\
 X \wedge I_+ & \xrightarrow{\quad} & M_f \xrightarrow{(x_1)} X
 \end{array}
 \qquad
 \begin{array}{l}
 M_f \times I' \hookrightarrow X \times I' \cup M_f \times \{0\} \\
 (y, t) \mapsto y \\
 (x, s, t) \mapsto
 \end{array}$$

Then, $X \rightarrow M_f$ is a cofibration, $M_f \rightarrow Y$ a homotopy equivalence.

Not hard to see using that M_f is a colimit.

proof of Whitehead's theorem. Consider $X \xrightarrow{f} M_f \rightarrow Y$ where $M_f \rightarrow Y$ is a homotopy equivalence. We know that $\pi_n(M_f, X) = 0$, and it suffices to see that this implies that M_f deformation retracts onto X .

By applying the lemma to $(X \cup Y, X) \hookrightarrow (M_f, X)$, we get a homotopy retraction of this to X , ~~$X \cup Y \rightarrow M_f$~~ Since $X \cup Y \rightarrow M_f$ is a cofibration (pushout), this extends to $M_f \rightarrow M_f$ taking $X \cup Y$ to X . Apply compression lemma again to

$$\underbrace{(X \times I_+ \cup Y, X \times \partial I_+ \cup Y)}_{\text{CW pair}} \rightarrow (M_f, X \cup Y) \rightarrow (M_f, X)$$

using pushout to get the homotopy.

$$\begin{array}{ccc}
 X \times I' \cup Y \times I' & & X \cup Y \rightarrow Z^{I'} \\
 \downarrow & \searrow & \downarrow \\
 M_f \times I' & \dashrightarrow & M_f \\
 \downarrow & & \downarrow \\
 M_f \times \{0\} & & Z
 \end{array}$$

Use retraction of I^2 onto $I \times \{0\} \cup \partial I \times I'$ to get a retraction of M_f onto $(X \cup Y) \times I' \cup M_f \times \{0\}$.