548.
Lecture 11.
6 February 2018.

__Thm__ (Whitehead, 1949). Suppose that X, Y are based connected CW-complexes. A weak homotopy equivalence $f: X \to Y$ is a homotopy equivalence.

__Compression Lemma__. (X,A) a CW pair, (Y,B) a pair with $B \neq \emptyset$. Assume that if $X-A$ has n-cells then $\pi_n(Y,B,y) = 0$ for all choices of y (meaning (Y,B) 0-connected if $n=0$). Then, every $f:(X,A) \to (Y,B)$ is homotopic rel A to a map $X \to B$.

__proof.__ ~~We can assume that X^0 maps via f to B~~

Assume that $h: [0, 1-\frac{1}{2^{k+1}}] \times X \to Y$ is a homotopy rel A from f to a map $X \to Y$ that takes X^k to B. Let $g = h|_{1-\frac{1}{2^{k+1}}}$. Then, for each $(k+1)$-cell of $X-A$, with attaching map $\phi_\alpha: S^k \to X$, we get a map

$$(D^{k+1}, S^k) \to (Y, B),$$

which is necessarily nullhomotopic since $\pi_{k+1}(Y,B,y)=0$. For each ϕ_α we get a homotopy rel S^k to $D^{k+1} \to B$. By definition of X^{k+1}, we get a homotopy from $g|_{X^{k+1}}$ to a map that sends X^{k+1} to B. By HEP for a CW pair, this homotopy extends to one on all of X. Place that along $[1-\frac{1}{2^{k+1}}, 1-\frac{1}{2^{k+2}}]$.

Relation of $\pi_n(Y,B)$ to homotopies rel S^{n-1}.

Given $D^n \times I' \longrightarrow Y$ mapping $S^{n-1} \times I$ to B, view $D^n \times I'$ as the discs

Then have the same boundary S^{n-1}, and we get a homotopy rel S^{n-1}.

Replacing a map with a cofibration.

Let ~~[illegible]~~ $X \xrightarrow{f} Y$ be any map, and consider the (pointed) mapping cylinder

$$\begin{tikzcd}
X \arrow[r] \arrow[d, shift right, "(x{,}0)"', "s"] & Y \arrow[d, shift right] & \\
X \wedge I_+ \arrow[r] \arrow[u, dashed, shift right] & M_*f \arrow[u, dashed, shift right] & X \arrow[l, <->, bend right, "(x{,}1)"']
\end{tikzcd}$$

$M_*f \times I' \longrightarrow X \times I' \cup M_*f \times \{0\}$

$(y,t) \longmapsto y$

$(x,s,t) \longmapsto$

Then, $X \to M_*f$ is a cofibration, $M_*f \to Y$ a homotopy equivalence.

Not hard to see using that M_*f is a colimit.

proof of Whitehead's theorem. Consider $X \to M_*f \to Y$ (with f over $X \to Y$) where $M_*f \to Y$ is a homotopy equivalence. We have that $\pi_n(M_*f, X) = 0$, and it suffices to see that this implies that M_*f deformation retracts onto X.

By applying the lemma to $(X \cup Y, X) \hookrightarrow (M_f, X)$, we get a homotopy rel X. of this to X, ~~[illegible] map $X \cup Y$~~ Since $X \cup Y \to M_f$ is a cofibration (pushout), this extends to a homotopy of id_{M_f} to $M_f \to M_f$ taking $X \cup Y$ to X.

Apply compression lemma again to

$$\underbrace{(X \wedge I_+ \vee Y,\ X \times \partial I \vee Y)}_{\text{CW pair}} \longrightarrow (M_f, X \cup Y) \longrightarrow (M_f, X),$$

using pushout to get the homotopy.

$$\begin{tikzcd}
X \times I' \cup Y \times I' \arrow[d] \arrow[dr] & \\
M_*f \times I' \arrow[r, dashed] & Z \\
M_*f \times \{0\} \arrow[u, hook] \arrow[ur] &
\end{tikzcd}
\qquad
\begin{tikzcd}
& X \cup Y \arrow[dl] \arrow[d] \arrow[r] & Z^{I'} \arrow[d] \\
X \times I' \cup Y \arrow[r] & M_*f \arrow[r] & Z
\end{tikzcd}$$

Use retraction of I^2 onto $I \times \{0\} \cup \partial I \times I'$ to get a retraction of M_*f onto $(X \cup Y) \times I' \cup M_*f \times \{0\}$.