CW-complexes.

We view spaces as being built up cell by cell. A CW-structure on a pointed space (X, x) consists of a family

$$X^n \subseteq X \qquad n \geq -1$$

of closed subspaces of X and for each $n \geq 1$ attaching maps

$$S^{n-1} \xrightarrow{\phi_\alpha} X^{n-1}, \qquad \alpha \in I_n$$

such that

(i) $X^{-1} = \{x\}$,

(ii) $X^0 \subseteq X$ is a discrete set containing x,

(iii) X^n is obtained from $X^{n-1}, \phi_\alpha, \alpha \in I_n$ as

$$X^{n-1} \cup_{S^{n-1}_\alpha} D^n_\alpha = X^{n-1} \cup_{\bigvee_{\alpha \in I_n} S^{n-1}} \left(\vee D^n\right),$$

(iv) $\bigcup_n X^n \to X$ is a homeomorphism.

The D^n are called cells in X.

Exs. (1) S^n built out of S^{n-1} by gluing two discs along identity maps.

(2) S^n built out of $X^0 = \{s\}$ by attaching D^n along the map S^{n-1} to s.

Def. A subcomplex $A \subseteq X$ is the union of cells. In particular, A is closed in X and is itself a CW complex. In this case, (X, A) is a CW pair.

Definition. A map of pointed spaces $i : A \to X$ is a cofibration if it satisfies the <u>homotopy extension property</u>: for g, h as below

$$\begin{tikzcd}
A \wedge I_+ \arrow[d, dash] \arrow[dr, "g"] & \\
X \wedge I_+ \arrow[r, dashed, "\exists h"] & Y \\
X \wedge \{0\}_+ \arrow[u, dash] \arrow[ur, "f"'] &
\end{tikzcd}$$

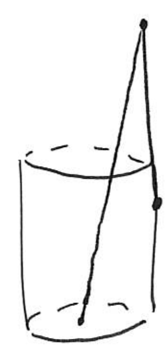
Perhaps present the unpointed version.

there exists h such that the diagram commute.

Lemma. An inclusion $i : A \hookrightarrow X$ is a cofibration if and only if $X \times I$ deformation retracts onto $A \wedge I_+ \cup X \wedge \{0\}_+$.

Lemma. If (X, A) is a CW pair, then $A \to X$ is a cofibration.

proof. $D^n \times \{0\} \cup \partial D^n \times I$ is a deformation retract of $D^n \times I$.



flow along these lines.

Hence, $S^{n-1} \subseteq D^n$ is a cofibration. Since pushouts of cofibrations are cofibrations and X is built from A out of these, $A \to X$ is a cofibration.