<u>Definitions</u>. (1) Let X, Y be topological spaces. Two maps $f, g : X \to Y$ are <u>homotopic</u> if there exists $h : X \times I \to Y$ such that $f = h|_{X \times \{0\}}$ and $g = h|_{X \times \{1\}}$. Diagrammatically,

$$\begin{tikzcd} X \arrow[d, "i_0"'] \arrow[dr, "f"] & \\ X \times I \arrow[r, dashed, "h"] & Y \\ X \arrow[u, "i_1"] \arrow[ur, "g"'] & \end{tikzcd}.$$

It's easy to see that this defines an equivalence relation on $\mathrm{Hom}_{\mathrm{Top}}(X, Y)$. We write

$$[X, Y] = \mathrm{Hom}_{\mathrm{Top}}(X, Y) / \text{homotopy}.$$

(2) If X, Y are pointed, by x, y, and if $f, g : (X, x) \to (Y, y)$ are pointed maps, then f and g [redacted] are <u>homotopic</u> if there exists $h : (X \wedge I_+, +) \to (Y, y)$ such that

$$h|_{X \wedge \{0\}_+} = f,$$

$$h|_{X \wedge \{1\}_+} = g.$$

This is a based homotopy. The homotopy itself must fix the basepoint. We set

$$[X, Y]_* = [(X, x), (Y, y)]_* = \mathrm{Hom}_{\mathrm{Top}_*}(X, Y) / \text{homotopy}.$$

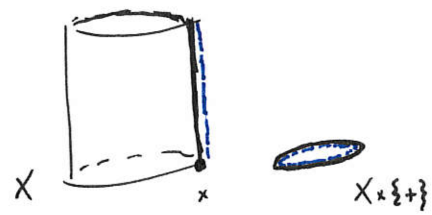
(3) $\quad X \wedge Y = X \times Y \big/ \overbrace{X\times\{y\} \cup \{x\}\times Y}^{X \vee Y}$, if x, y are the basepoints of X, Y. This is called the <u>smash</u> <u>product</u> of X, Y. It replaces the product $X \times Y$ in most (all?) based constructions.

<u>Ex.</u> $\quad X \wedge S^0 \cong X$.

$\cong$: homeomorphism.

This means that S^0 is the <u>unit</u> for $\wedge$.

<u>Ex.</u>



So, $\quad X \wedge I_+ \cong X \times I / \{x\}\times I$.

Think about mapping out of $X \wedge I_+$. This is determined by mapping out of $X \times I$ so that $\{x\}\times I$ is collapsed to the basepoint of (Y, y). Hence, definition (2) above agrees with what you might have expected.

Definition. Let $S^n \subset \mathbb{R}^{n+1}$ be the space of solutions to $x_0^2 + \cdots + x_n^2 = 1$, pointed by $s = (1, 0, \ldots, 0)$. If (X, x) is a pointed space,

$$\pi_n(X, x) = [S^n, (X, x)]_* = [S^n, X]_*,$$

the nth homotopy group of (X, x) (or just X).

We will get into the properties of $\pi_n(X, x)$ next lecture. In particular, we will see why these are mostly abelian groups.

Exs. (1) $\pi_0(X, x)$ is the set of path-components of X, pointed by the component containing x.

(2) $\pi_1(S^1) \cong \mathbb{Z}$.

(3) $\pi_n(S^1) = 0$ for $n > 1$. Indeed, recall that $\mathbb{R} \to S^1$ is a covering space. If $n > 1$, there is a lift

$$\begin{tikzcd} & \mathbb{R} \arrow[d] \\ S^n \arrow[r] \arrow[ur, dashed] & S^1 \end{tikzcd}.$$

But, $\pi_n(\mathbb{R}) = 0$ for all n.

Hopf invariants. Let $f: S^{2n-1} \to S^n$ be a (continuous) map, $n > 1$. Let C_f be the cone, i.e., the CW complex obtained from S^n by attaching D^{2n} along f. Then,

$$H^*(C_f, \mathbb{Z}) = \begin{cases} \mathbb{Z} & * = 0, n, 2n, \\ 0 & \text{otherwise}. \end{cases}$$

Let α generate $H^n(C_f, \mathbb{Z})$ and β generate $H^{2n}(C_f, \mathbb{Z})$. Then, $\alpha^2 = H(f)\beta$, where $H(f)$ is the Hopf invariant.

These generators should be compatible with the cell structure, so that α pulls back to the positive generator of $H^n(S^n, \mathbb{Z})$, while β corresponds to D^{2n} in cellular cohomology.

This gives a homomorphism $\pi_{2n-1}(S^n) \to \mathbb{Z}$, whose image always includes $2\mathbb{Z}$. when n is even When is there a map of Hopf invariant 1?

Thm (Adams). If and only if $n = 1, 2, 4, 8$. These are exactly the dimensions where $\mathbb{R}^n$ admits a (continuous) division algebra structure, $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{O}$. These are also the only dimensions in which S^{n-1} is parallelizable, i.e., when S^n admits n linearly independent vector fields.

This turns out to be closely related to counting differentiable manifold structures on S^{n-1}! There are 28 on S^7, the first place where exotic smooth structures appear.

We will also see that $H(f)$ can be defined as an integral on $S^{n-1} \times S^{n-1}$ and as the linking number of two disjoint $(n-1)$-spheres in $\mathbb{R}^{2n-1}$. When $n=2$, this is closely related to knot theory.

Ex. In n is odd, $H(f) = 0$. This follows from graded commutativity of the cup product.