Problem set 5 for 131 A/3 - Fall 2012

Benjamin Antieau

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1. Let $f : [0,1] \to [0,1]$ be a continuous function. Prove that $f(x) = x$ for at least one $x \in [0,1]$ [Rud87].

2. [Ros80, Exercise 18.5]. Let f and g be continuous functions on $[a,b]$ such that $f(a) \geq g(a)$ and $f(b) \leq g(b)$. Prove that $f(x_0) = g(x_0)$ for at least one $x_0 \in [a,b]$.

3. Show that if f is an odd-degree polynomial function, then f has at least one real root.

4. Prove that if f is a real uniformly continuous function on the bounded set E of $\mathbb{R}$, then f is bounded on E. Show that this is false if f is only continuous [Rud87].

5. [Ros80, Exercise 19.1].

6. [Ros80, Exercise 19.8]. Prove that $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$. Show that $\sin x$ is uniformly continuous on $\mathbb{R}$.

7. [Ros80, Exercise 19.9]. Let $f(x) = x\sin(1/x)$ for $x \neq 0$ and $f(0) = 0$. Is f uniformly continuous on $\mathbb{R}$?

8. Prove that the composition of two uniformly continuous functions is uniformly continuous [Rud87].

9. Let $f(x)$ be a function possessing the property that to every x_0 there corresponds a $\delta > 0$ such that $f(x) \geq f(x_0)$ whenever $|x - x_0| < \delta$. Prove that the set of values of $f(x)$ is finite or countable [Nat55].

10. [Ros80, Exercise 20.1].

References

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[Nat55] I. P. Natanson, *Theory of functions of a real variable*, Frederick Ungar Publishing Co., New York, 1955. Translated by Leo F. Boron with the collaboration of Edwin Hewitt.

[Ros80] K. A. Ross, *Elementary analysis: the theory of calculus*, Springer-Verlag, New York, 1980. Undergraduate Texts in Mathematics.

[Rud87] W. Rudin, *Real and complex analysis*, 3rd ed., McGraw-Hill Book Co., New York, 1987.