061 - Midterm 2 - Practice Problems

14 May 2011

1. Solve the recurrence relation

$$a_n = 5a_{n-1} - 6a_{n-2}$$

subject to the initial conditions $a_0 = 5$ and $a_1 = 4$.

Solution First, solve $x^2 = 5x - 6$. We get $x^2 - 5x + 6 = (x - 3)(x - 2)$. Thus, we search for a solution to

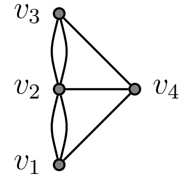
$$a_n = b \cdot 3^n + c \cdot 2^n.$$

Plugging in $n = 0$, we have $5 = b + c$. Plugging in $n = 1$, we have $4 = 3b + 2c$. Thus, $b = -6$ and $c = 11$. So, the solution is $a_n = -6 \cdot 3^n + 11 \cdot 2^n$.

2. Let a_n be the number of strings of length n on the set $X = \{0, 1, 2\}$ with no consecutive 0s. Find a linear homogeneous recurrence relation of order 2 satisfied by a_n.

Solution Let Y_n denote the set of strings of length n on the set X with no consecutive 0s. Let $W_n \subseteq Y_n$ be the subset consisting of strings not starting with 0, and let $Z_n \subseteq Y_n$ be the subset of strings starting with 0. Thus, $W_n \cap Z_n = \emptyset$ and $Y_n = W_n \cup Z_n$. So, $a_n = |Y_n| = |W_n| + |Z_n|$. If $s \in W_n$, then $s = 1t$ or $s = 2t$ where t is a string of length $n - 1$ on X with no consecutive 0s. Thus, $|W_n| = 2a_{n-1}$. On the other hand, if $s \in Z_n$, then $s = 01t$ or $s = 02t$, where t is a string of length $n - 2$ with no consecutive 0s. So, $|Z_n| = 2a_{n-2}$. Therefore, $a_n = 2a_{n-1} + 2a_{n-2}$.

3. Let K be the graph



How many paths of length 3 from v_1 to v_2 are there?

Solution Consider the adjacency matrix of K:

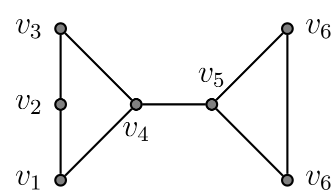
$$A = \begin{pmatrix} 0 & 2 & 0 & 1 \\ 2 & 0 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{pmatrix}.$$

We need to compute the $(1,2)$-entry of A^3. We find by matrix multiplication that the matrix A^3 is

$$A = \begin{pmatrix} 4 & 22 & 4 & 11 \\ 22 & 8 & 22 & 11 \\ 2 & 22 & 4 & 11 \\ 11 & 11 & 11 & 8 \end{pmatrix}.$$

So, there are 22 paths of length 3 from v_1 to v_2 in K.

4. Prove that the graph G



does not have a Hamiltonian cycle.

Solution Suppose that there was a Hamiltonian cycle P in G. We may assume that the starting and ending point of the cycle is v_1. Then, since P visits every vertex, at some point, P visits v_4 and then v_5. Later on, it visits v_5 and v_4. Since it visits v_4 more than once, it cannot be a Hamiltonian cycle, so G has no Hamiltonian cycle.

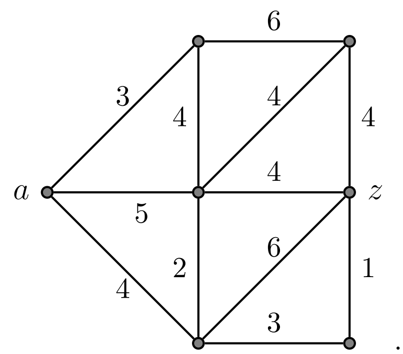
5. Prove that there is no simple graph with 5 vertices such that the degree of every vertex is 3.

Solution Suppose that G is such a graph. Then, the total degree of G is 15. But, this is also twice the number of edges, and so an even number, a contradiction.

6. Which complete bipartite graphs $K_{m,n}$ with $m > 0$ and $n > 0$ have Euler cycles?

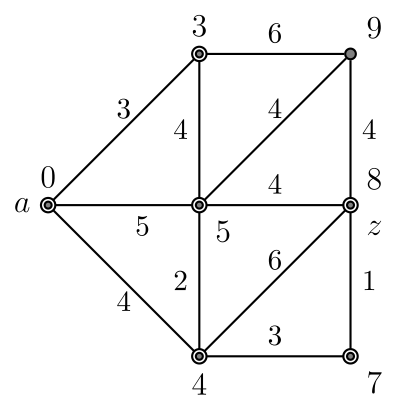
Solution A connected graph has an Euler cycle if and only if the degree of every vertex is even. The vertices of $K_{m,n}$ are split into two sets, A and B, where there are m elements in A and n elements in B. The degree of every element of A is n and the degree of every element of B is m. Thus, $K_{m,n}$ has an Euler cycle if and only if both m and n are even.

7. Consider the weighted graph G



Run Dijkstra's algorithm to find the length of the shortest path from a to z. Draw the state of the graph when the algorithm finishes together with all labels permanent or not on all vertices.

Solution



8. Let G be a simple graph with 11 vertices. Show that either G or its complement $\overline{G}$ is not planar.

Solution If G is not planar, we're done. So, suppose that G is a planar graph on 11 vertices. The total number of edges of G and $\overline{G}$ is the number of edges in K_{11}, which is 55. Now, note that if H is a simple planar graph, then $2e \geq 3f = 3(e-v+2)$. Thus, $e \leq 3v-6$. In our case, $v = 11$, so we obtain the inequality $e \leq 33-6 = 27$. Thus, since G is planar, $e \leq 27$. Therefore, since the number of edges in G plus the number of edges in $\overline{G}$ is 55, it follows that $\overline{G}$ has at least 28 edges. By the inequality $e \leq 27$ for planar graphs on 11 vertices, we see that $\overline{G}$ cannot be planar.