characteristic polynomial of T_A splits into factors of the form $t-\lambda$. Since each λ is real, the characteristic polynomial splits over R. But T_A has the same characteristic polynomial as A, which has the same characteristic polynomial as T. Therefore the characteristic polynomial of T splits. ∎

HW 9.

We are now able to establish one of the major results of this chapter.

Theorem 6.17. *Let* T *be a linear operator on a finite-dimensional real inner product space* V. *Then* T *is self-adjoint if and only if there exists an orthonormal basis* β *for* V *consisting of eigenvectors of* T.

Proof. Suppose that T is self-adjoint. By the lemma, we may apply Schur's theorem to obtain an orthonormal basis β for V such that the matrix $A = [\mathsf{T}]_\beta$ is upper triangular. But

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$$A^* = [\mathsf{T}]_\beta^* = [\mathsf{T}^*]_\beta = [\mathsf{T}]_\beta = A.$$

So A and A^* are both upper triangular, and therefore A is a diagonal matrix. Thus β must consist of eigenvectors of T.

The converse is left as an exercise. ∎

Theorem 6.17 is used extensively in many areas of mathematics and statistics. We restate this theorem in matrix form in the next section.

Example 4

As we noted earlier, real symmetric matrices are self-adjoint, and self-adjoint matrices are normal. The following matrix A is complex and symmetric:

$$A = \begin{pmatrix} i & i \\ i & 1 \end{pmatrix} \quad \text{and} \quad A^* = \begin{pmatrix} -i & -i \\ -i & 1 \end{pmatrix}.$$

But A is not normal, because $(AA^*)_{12} = 1+i$ and $(A^*A)_{12} = 1-i$. Therefore complex symmetric matrices need not be normal. ◆

EXERCISES

1. Label the following statements as true or false. Assume that the underlying inner product spaces are finite-dimensional.
   (a) Every self-adjoint operator is normal.
   (b) Operators and their adjoints have the same eigenvectors.
   (c) If T is an operator on an inner product space V, then T is normal if and only if $[\mathsf{T}]_\beta$ is normal, where β is any ordered basis for V.
   (d) A real or complex matrix A is normal if and only if L_A is normal.
   (e) The eigenvalues of a self-adjoint operator must all be real.
   (f) The identity and zero operators are self-adjoint.
   (g) Every normal operator is diagonalizable.
   (h) Every self-adjoint operator is diagonalizable.

2. For each linear operator T on an inner product space V, determine whether T is normal, self-adjoint, or neither. If possible, produce an orthonormal basis of eigenvectors of T for V and list the corresponding eigenvalues.
   (a) $\mathsf{V} = \mathsf{R}^2$ and T is defined by $\mathsf{T}(a,b) = (2a-2b, -2a+5b)$.
   (b) $\mathsf{V} = \mathsf{R}^3$ and T is defined by $\mathsf{T}(a,b,c) = (-a+b, 5b, 4a-2b+5c)$.
   (c) $\mathsf{V} = \mathsf{C}^2$ and T is defined by $\mathsf{T}(a,b) = (2a+ib, a+2b)$.
   (d) $\mathsf{V} = \mathsf{P}_2(R)$ and T is defined by $\mathsf{T}(f) = f'$, where
   $$\langle f, g\rangle = \int_0^1 f(t)g(t)\,dt.$$
   (e) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$ and T is defined by $\mathsf{T}(A) = A^t$.
   (f) $\mathsf{V} = \mathsf{M}_{2\times 2}(R)$ and T is defined by $\mathsf{T}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$.
3. Give an example of a linear operator T on R^2 and an ordered basis for R^2 that provides a counterexample to the statement in Exercise 1(c).
4. Let T and U be self-adjoint operators on an inner product space V. Prove that TU is self-adjoint if and only if $\mathsf{TU} = \mathsf{UT}$.
5. Prove (b) of Theorem 6.15.
6. Let V be a complex inner product space, and let T be a linear operator on V. Define
   $$\mathsf{T}_1 = \frac{1}{2}(\mathsf{T}+\mathsf{T}^*) \quad \text{and} \quad \mathsf{T}_2 = \frac{1}{2i}(\mathsf{T}-\mathsf{T}^*).$$
   (a) Prove that T_1 and T_2 are self-adjoint and that $\mathsf{T} = \mathsf{T}_1 + i\,\mathsf{T}_2$.
   (b) Suppose also that $\mathsf{T} = \mathsf{U}_1 + i\mathsf{U}_2$, where U_1 and U_2 are self-adjoint. Prove that $\mathsf{U}_1 = \mathsf{T}_1$ and $\mathsf{U}_2 = \mathsf{T}_2$.
   (c) Prove that T is normal if and only if $\mathsf{T}_1\mathsf{T}_2 = \mathsf{T}_2\mathsf{T}_1$.
7. Let T be a linear operator on an inner product space V, and let W be a T-invariant subspace of V. Prove the following results.
   (a) If T is self-adjoint, then T_W is self-adjoint.
   (b) $\mathsf{W}^\perp$ is T^*-invariant.
   (c) If W is both T- and T^*-invariant, then $(\mathsf{T}_\mathsf{W})^* = (\mathsf{T}^*)_\mathsf{W}$.
   (d) If W is both T- and T^*-invariant and T is normal, then T_W is normal.

8. Let T be a normal operator on a finite-dimensional complex inner product space V, and let W be a subspace of V. Prove that if W is T-invariant, then W is also T*-invariant. *Hint:* Use Exercise 24 of Section 5.4.

9. Let T be a normal operator on a finite-dimensional inner product space V. Prove that $N(T) = N(T^*)$ and $R(T) = R(T^*)$. *Hint:* Use Theorem 6.15 and Exercise 12 of Section 6.3.

10. Let T be a self-adjoint operator on a finite-dimensional inner product space V. Prove that for all $x \in V$

$$\|T(x) \pm ix\|^2 = \|T(x)\|^2 + \|x\|^2.$$

Deduce that $T - iI$ is invertible and that $[(T - iI)^{-1}]^* = (T + iI)^{-1}$.

11. Assume that T is a linear operator on a complex (not necessarily finite-dimensional) inner product space V with an adjoint T*. Prove the following results.
    (a) If T is self-adjoint, then $\langle T(x), x\rangle$ is real for all $x \in V$.
    (b) If T satisfies $\langle T(x), x\rangle = 0$ for all $x \in V$, then $T = T_0$. *Hint:* Replace x by $x + y$ and then by $x + iy$, and expand the resulting inner products.
    (c) If $\langle T(x), x\rangle$ is real for all $x \in V$, then $T = T^*$.

12. Let T be a normal operator on a finite-dimensional real inner product space V whose characteristic polynomial splits. Prove that V has an orthonormal basis of eigenvectors of T. Hence prove that T is self-adjoint.

13. An $n \times n$ real matrix A is said to be a **Gramian** matrix if there exists a real (square) matrix B such that $A = B^tB$. Prove that A is a Gramian matrix if and only if A is symmetric and all of its eigenvalues are nonnegative. *Hint:* Apply Theorem 6.17 to $T = L_A$ to obtain an orthonormal basis $\{v_1, v_2, \ldots, v_n\}$ of eigenvectors with the associated eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. Define the linear operator U by $U(v_i) = \sqrt{\lambda_i}v_i$.

14. *Simultaneous Diagonalization.* Let V be a finite-dimensional real inner product space, and let U and T be self-adjoint linear operators on V such that $UT = TU$. Prove that there exists an orthonormal basis for V consisting of vectors that are eigenvectors of both U and T. (The complex version of this result appears as Exercise 10 of Section 6.6.) *Hint:* For any eigenspace $W = E_\lambda$ of T, we have that W is both T- and U-invariant. By Exercise 7, we have that $W^\perp$ is both T- and U-invariant. Apply Theorem 6.17 and Theorem 6.6 (p. 350).

15. Let A and B be symmetric $n \times n$ matrices such that $AB = BA$. Use Exercise 14 to prove that there exists an orthogonal matrix P such that P^tAP and P^tBP are both diagonal matrices.

16. Prove the *Cayley–Hamilton theorem* for a complex $n \times n$ matrix A. That is, if $f(t)$ is the characteristic polynomial of A, prove that $f(A) = O$. *Hint:* Use Schur's theorem to show that A may be assumed to be upper triangular, in which case

$$f(t) = \prod_{i=1}^{n}(A_{ii} - t).$$

Now if $T = L_A$, we have $(A_{jj}I - T)(e_j) \in \text{span}(\{e_1, e_2, \ldots, e_{j-1}\})$ for $j \geq 2$, where $\{e_1, e_2, \ldots, e_n\}$ is the standard ordered basis for C^n. (The general case is proved in Section 5.4.)

The following definitions are used in Exercises 17 through 23.

Definitions. *A linear operator* T *on a finite-dimensional inner product space is called* ***positive definite*** *[****positive semidefinite****] if* T *is self-adjoint and* $\langle T(x), x\rangle > 0$ $[\langle T(x), x\rangle \geq 0]$ *for all* $x \neq 0$.

An $n \times n$ *matrix* A *with entries from* R *or* C *is called* ***positive definite*** *[****positive semidefinite****] if* L_A *is positive definite [positive semidefinite].*

17. Let T and U be a self-adjoint linear operators on an n-dimensional inner product space V, and let $A = [T]_\beta$, where β is an orthonormal basis for V. Prove the following results.
    (a) T is positive definite [semidefinite] if and only if all of its eigenvalues are positive [nonnegative].
    (b) T is positive definite if and only if

$$\sum_{i,j} A_{ij}a_j\bar{a}_i > 0 \text{ for all nonzero } n\text{-tuples } (a_1, a_2, \ldots, a_n).$$

    (c) T is positive semidefinite if and only if $A = B^*B$ for some square matrix B.
    (d) If T and U are positive semidefinite operators such that $T^2 = U^2$, then $T = U$.
    (e) If T and U are positive definite operators such that $TU = UT$, then TU is positive definite.
    (f) T is positive definite [semidefinite] if and only if A is positive definite [semidefinite].

Because of (f), results analogous to items (a) through (d) hold for matrices as well as operators.