115a/4 - Homework 6*

Due 5 November 2010

1 Direct sums

Definition 1.1. Suppose that V is a vector space and that W and Z are two subspaces of V. Then, V is the sum of W and Z if every vector v in V may be written $v = w + z$ for some vector w in W and z in Z. Write $V = W + Z$.

Definition 1.2. If a vector space V is the sum of two subspaces W and Z, say that the sum is direct if every element v of V may be written uniquely as $v = w + z$. In other words, if $v = w' + z'$, then $w' = w$ and $z' = z$. Say that V is the direct sum of W and Z, and write $V = W \oplus Z$.

1. Show that if $V = W + Z$, then $V = W \oplus Z$ if and only if

$$W \cap Z = (0),$$

where $W \cap Z$ is the intersection of the two subspaces W and Z.

For the next three problems, assume that V is finite dimensional, that $V = W \oplus Z$, that $T : V \to V$ is a linear operator, and that W and Z are both T-invariant.

2. Show that there is some basis β for V such that the matrix representation of T with respect to β can be written as a block-diagonal matrix

$$[T]_\beta = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}.$$

3. Show that the linear operator T restricts to linear operators $T|_W : W \to W$ and $T|_Z : Z \to Z$.

4. Show that

$$det(T) = det(T|_W)det(T|_Z).$$

*Numbers in parentheses like (1.2.11) refer to the 11th problem in the second section of the first chapter of Friedberg *et. al.*

2 Characteristic

Definition 2.1. For any field F, any element a of F, and any positive integer n, the element $na \in F$ is defined as

$$na = a + \cdots a,$$

the n-fold sum of a with itself.

Definition 2.2. The characteristic of the field F is defined to be the smallest positive integer n such that $na = 0$ for all $a \in F$. If no such integer exists, say that the characteristic is 0.

5. Show that the characteristic of a field is either 0 or a prime number.

Definition 2.3. A field homomorphism is a function $i : F \to G$, where F and G are fields, such that $i(1) = 1$, $i(a + b) = i(a) + i(b)$, and $i(ab) = i(a)i(b)$.

6. Show that the map $\mathbb{R} \to \mathbb{C}$ defined by sending a to $a + 0i$ is a field homomorphism.

7. Show that if F is a characteristic p field, where $p > 0$, then the map that takes a to a^p is a field homomorphism from F to itself.

8. Suppose that F is a characteristic p field, where $p > 0$. Compute the nullity and rank of the differentiation map

$$\frac{d}{dx} : P_p(F) \to P_{p-1}(F).$$

3 One-sided inverses

Definition 3.1. Let $T : V \to W$ be a linear transformation. A map $S : W \to V$ is a called a left inverse to T is $S \circ T = Id_V$. A map $S : W \to V$ is called a right inverse to T if $T \circ S = Id_W$.

9. Show that the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ sending (a, b) to $(a, b, 0)$ has a left inverse but not a right inverse.

10. Show that T has a left inverse if and only if T is one-to-one. Show that T has a right inverse if and only if T is onto.

11. Show that if T has a left and a right inverse then they are the same. Conclude that if this is the case, then T is invertible.